arXiv:gr-qc/0005047 13 May 2000

Time-Independent Gravitational Fields

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1 Introduction

In this article we want to describe what is known about time independent spacetimes from a global point of view. The physical situations we want to treat are isolated bodies at rest or in uniform rotation in an otherwise empty universe. In such cases one expects the gravitational field to have no "independent degrees of freedom". Very loosely speaking, the spacetime geometry should be uniquely determined by the matter content of the model under consideration. In a similar way, for a given matter model (such as that of a perfect fluid), there should be a one-to-one correspondence between Newtonian solutions and general relativistic ones.

The plan of this paper is as follows. In Sect. 2.1 we collect the information afforded on one hand by the Killing equation obeyed by the vector field generating the stationary isometry and, on the other hand, that by the Einstein field equations. Throughout this paper we assume this stationary isometry to be everywhere timelike. Thus ergoregions are excluded. We try as much as possible to write the resulting equations in terms of objects intrinsic to the space (henceforth simply called "the quotient space") obtained by quotienting spacetime by the action of the stationary isometry. Much of this is standard. But since none of the references known to us meets our specific purposes, we give a self-contained treatment starting from scratch. Since all the models we treat are axially symmetric, we add in Sect. 2.2 a second, axial Killing vector to the formalism of Sect. 2.1.

In Sect. 2.3 we first introduce new dependent variables for the vacuum gravitational field, namely a conformally rescaled metric on the quotient space and two potentials, using which the field equations have an interpretation in terms of harmonic maps from the quotient space into the Poincaré half plane. These potentials, originally due to Hansen, are used in our treatment of asymptotics in Sects. 3.1,2. We then formulate the boundary conditions at spatial infinity appropriate for isolated systems and prove two basic theorems due to Lichnerowicz on stationary solutions obeying these conditions. These theorems are manifestations of the above-mentioned principle concerning the lack of gravitational degrees of freedom. The first result, the "staticity theorem", basically states that the gravitational field is static when the matter is non-rotating. The second one, the "vacuum theorem", states that spacetime is Minkowski when no matter is present. In Sect. 2.4 we restore c, the velocity of light, in the field equations and show that these tend to the Newtonian ones as $c \to \infty$.

In Chapter 3 we we study solutions only "near infinity". (Note that by the Lichnerowicz vacuum theorem, such solutions can not be extended to all of $\mathbf{R^4}$ exept for flat spacetime.) In the Newtonian case such solutions are known to have a convergent expansion in negative powers of the radius where the coefficients are given by multipole moments. The relativistic situation is slightly at variance with our statement at the beginning concerning the Newton–Einstein correspondence: namely, there are now, corresponding to the presence of two potentials rather than one, two infinite sequences of multipole moments, the "mass moments" which have a Newtonian analogue and the "angular momentum moments" which do not. One may now study the two potentials and the rescaled quotient space metric in increasing powers of $1/r$, where r is the radius corresponding to a specific coordinate gauge on the quotient space which has to be readjusted at each order in $1/r$.

The results one finds are sufficient for the existence of a chart in the one-point "compactification" of the quotient space (i.e. the union of the quotient space and the point-at-infinity), in terms of which yet another conformal rescaling of the 3-metric, together with a corresponding rescaling of the two potentials, admit regular extensions to the compactified space. As summarized in Sect. 3.2 one is then able to find field equations for these "unphysical" variables which are regular at the point-at-infinity and in addition can be turned into an elliptic system. From this it follows that the unphysical quantities are in fact analytic near infinity and this, in turn, implies convergence for a suitable $1/r$-expansion (r being a "physical" radius) for the original physical variables. Furthermore the structure of the unphysical equations yields the result that the (physical) spacetime metric is uniquely characterized by the two sets of multipole moments.

It is remarkable that stationary vacuum solutions satisfying rather weak fall–off conditions at spatial infinity, by the very nature of the field equations, have to have a convergent multipole expansion. We believe that the topic of far–field behaviour of time–independent gravitational fields is by now reasonably well understood. The main open problem is to characterize an a priori given sequence of multipole moments for which the expansion converges.

In Chapter 4 we review global rotating solutions. In Sect. 4.1 we outline a result due to Lindblom which shows that stationary rotating spacetimes with a one-component fluid source with phenomenological heat conduction and viscosity have to be axisymmetric. In Sect. 4.2 we describe a theorem of Heilig which proves the existence of axisymmetric, rigidly rotating perfect-fluid spacetimes with polytropic equation of state, provided the parameters are sufficiently close to ones for a nonrotating Newtonian solution. In Sect. 4.3 we present the solution of Neugebauer and Meinel representing a rigidly rotating infinitely thin disk of dust. In the final chapter we treat global nonrotating solutions.

In Sect. 5.1 we outline the essentials of a relativistic theory of static elastic bodies. The remaining sections are devoted to spherical symmetry. It has long been conjectured that nonrotating perfect fluids are spherical whence Schwarzschild in their exterior region. In Sect. 5.2 we discuss the present status of this conjecture. A proof exists when the allowed equations of state are limited by a certain inequality. While this inequality covers many cases of physical interest, the Newtonian situation suggests that the conjecture is probably true without this restriction. In Sect. 5.3 we review spherically symmetric perfect fluid solutions. The final Sect. 5.4 gives a short description of self–gravitating Vlasov matter in the sperically symmetric case.

In the subject of time–independent gravitational field of isolated bodies there are some topics we do not cover. We do not address the question of the conjectured non–existence of solutions with more than one body. (Müller zum Hagen [58] has some results on this in the static case.) Furthermore we limit ourselves to "standard matter" sources. Thus Black Holes are excluded. (For this see the article of Maison in this volume.) We also could not cover the interesting case of soliton–like solutions for "non–linear matter sources", starting with the discovery of the Bartnik–McKinnon solutions of the Einstein–Yang Mills system (see Bizon [11].)

Acknowledgements

Part of this work was carried out at the Institute for Theoretical Physics at Santa Barbara, where the authors took part in the program on "Classical and Quantum Physics of Strong Gravitational Fields". We thank the ITP for its support and kind hospitality. R. Beig was in part supported by "Fonds zur Förderung der wissenschaftlichen Forschung in Österreich", project P12626-PHY.

2 Field Equations

2.1 Generalities

Let $(M, g_{\mu\nu})$ be a 4-dimensional smooth connected manifold with Lorentz metric $g_{\mu\nu}$ of signature $(- + ++)$. We assume M to be chronological, i.e. to admit no closed timelike curves. Let ξ^μ be an everywhere timelike Killing vector field with complete orbits. Thus we do not allow points where ξ^μ turns null, i.e. we exclude horizons and ergospheres. It follows (see [26]) that the quotient of M by the isometry group generated by ξ^μ is a Hausdorff manifold N and that M is a principal $\mathbf{R}^1$-bundle over N. Furthermore this bundle is trivial, i.e. M is diffeomorphic to $\mathbf{R}^1 \times N$. The fact that this diffeomorphism is non-natural (whereas of course the projection π mapping M onto N is) plays a role in the formalism we shall now develop.

Let us introduce the differential geometric machinery necessary for writing the stationary Einstein equations in a way naturally adapted to ξ^μ. As far as possible, we will be interested in quantities and equations intrinsic to N ("dimensional reduction"). For a similar treatment see the Appendix of [23]. We define the fields V and $\omega_{\lambda\nu\lambda} = \omega_{[\mu\nu\lambda]}$ by

$$V \quad := \xi_\mu \xi^\mu \Rightarrow V < 0 \tag{2.1}$$

$$\omega_{\mu\nu\lambda} := 3\xi_{[\mu}\nabla_\nu \xi_{\lambda]}. \tag{2.2}$$

The 3-form $\omega_{\mu\nu\lambda}$ vanishes if and only if ξ^μ is hypersurface orthogonal – in which case $(M, g_{\mu\nu})$ is called static. More important than $\omega_{\mu\nu\lambda}$ will be the 2-form $\sigma_{\mu\nu}$, given by

$$\sigma_{\mu\nu} := \omega_{\mu\nu\lambda}\xi^\lambda. \tag{2.3}$$

Given ξ^μ, the fields $\sigma_{\mu\nu}$ and $\omega_{\mu\nu\lambda}$ carry the same information, since

$$\omega_{\mu\nu\lambda} = 3V^{-1}\xi_{[\mu}\sigma_{\nu\lambda]}. \tag{2.4}$$

Equ. (2.4) is obtained by expanding the identity $\xi_{[\mu}\omega_{\nu\lambda\rho]} = 0$, which follows from (2.2), and contracting with ξ^μ. In a similar way we obtain the relations

$$\omega_{\mu\nu\lambda}\omega^{\mu\nu\lambda} = 3V^{-1}\sigma_{\mu\nu}\sigma^{\mu\nu} \tag{2.5}$$

$$\omega_{\mu\nu\lambda}\sigma^{\nu\lambda} = \frac{1}{3}\omega_{\rho\nu\lambda}\omega^{\rho\nu\lambda}\xi_\mu. \tag{2.6}$$

We now invoke the Killing equation for ξ^μ, i.e.

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.7}$$

Expanding $\omega_{\mu\nu\lambda}$ in terms of ξ_μ, we easily see that

$$\nabla_\mu \xi_\nu = V^{-1}[\sigma_{\mu\nu} + (\nabla_{[\mu}V)\xi_{\nu]}], \tag{2.8}$$

or, equivalently,

$$\sigma_{\mu\nu} = V^2 \nabla_{[\mu}(V^{-1}\xi_{\nu]}). \tag{2.9}$$

In the static case we have $\sigma_{\mu\nu} = 0$, whence there exist global cross sections given by $t = \text{const}$, where $\xi_\mu = V\nabla_\mu t$.

Equ. (2.9) implies that

$$\nabla_{[\mu}(V^{-2}\sigma_{\nu\lambda]}) = 0. \tag{2.10}$$

Clearly we have $\mathcal{L}_\xi \tau = 0$, where τ is the 3-form given by $\tau_{\mu\nu\lambda} = V^{-2}\omega_{\mu\nu\lambda}$. By (2.10) and the identity $\mathcal{L}_\xi = \xi \rfloor d\tau + d(\xi \rfloor \tau)$, this implies $\xi \rfloor d\tau = 0$. Since $d\tau$ is a 4-form and ξ^μ is nowhere zero, we infer in 4 dimensions that $d\tau$ is zero, i.e.

$$\nabla_{[\mu}(V^{-2}\omega_{\nu\lambda\rho]}) = 0. \tag{2.11}$$

Equ.'s (2.10,11) are integrability conditions for the Killing equations (2.7) which are "purely geometric" in that they do not involve the Ricci (whence: energy-momentum) tensor. Now recall the relation

$$\nabla_\mu \nabla_\nu \xi_\lambda = -R_{\nu\lambda\mu}{}^\rho \xi_\rho, \tag{2.12}$$

which follows from (2.7) and its corollary

$$g^{\nu\rho} \nabla_\nu \nabla_\rho \xi_\mu = -R_\mu{}^\nu \xi_\nu. \tag{2.13}$$

From (2.2), (2.7) and (2.13) we find that

$$\nabla^\mu \omega_{\mu\nu\lambda} = 2\xi_{[\nu} R_{\lambda]\mu} \xi^\mu, \tag{2.14}$$

which, using (2.8), implies

$$\nabla^\mu (V^{-1} \sigma_{\mu\nu}) = 2V^{-1} \xi_{[\nu} R_{\lambda]\mu} \xi^\mu \xi^\lambda - V^{-3} \sigma_{\mu\lambda} \sigma^{\mu\lambda} \xi_\nu, \tag{2.15}$$

where we have also used (2.6,7). Interpreting $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ as the energy-momentum tensor of matter, the r.h. side of Equ. (2.14) is zero iff the matter current, for an observer at rest relative to ξ^μ, is zero. In that case, and provided that M is simply connected, there exists a scalar field ω, called twist potential, such that

$$\omega_{\mu\nu\lambda} = \frac{1}{2} \varepsilon_{\mu\nu\lambda}{}^\rho \nabla_\rho \omega, \tag{2.16}$$

and then (2.11) implies

$$\nabla^\mu (V^{-2} \nabla_\mu \omega) = 0. \tag{2.17}$$

Note that, by virtue of $\xi_{[\mu} \omega_{\nu\lambda\rho]} = 0$, ω satisfies $\mathcal{L}_\xi \omega = 0$.

Next, using the definition (2.1) and Equ. (2.8), it is straightforward to show that

$$\begin{aligned} \nabla_\mu \nabla_\nu V = -2R_{\mu\lambda\nu\rho} \xi^\lambda \xi^\rho + 2V^{-2} [&\sigma_{\mu\lambda} \sigma_\nu{}^\lambda - (\nabla^\lambda V) \xi_{(\mu} \sigma_{\nu)\lambda} \\ &+ \frac{1}{4} V \nabla_\mu V \nabla_\nu V + \frac{1}{4} \xi_\mu \xi_\nu (\nabla V)^2] \end{aligned} \tag{2.18}$$

and whence

$$\nabla^\mu \nabla_\mu V = -2R_{\mu\nu} \xi^\mu \xi^\nu + V^{-1} (\nabla V)^2 + 2V^{-2} \sigma_{\mu\nu} \sigma^{\mu\nu}. \tag{2.19}$$

Now recall (see e.g. the Appendix of [23]) that there is a $1-1$ correspondence between tensor fields on M with vanishing Lie derivative with respect to ξ^μ and such that all their contractions with ξ^μ and ξ_μ are zero – and ones of the same type on N. In the case of covariant tensor fields on N, this correspondence is the same as pull-back under π. Examples of such tensor fields on M are the scalar field V, the symmetric tensor field

$$h_{\mu\nu} := g_{\mu\nu} - V^{-1} \xi_\mu \xi_\nu \tag{2.20}$$

and the 2-form $\sigma_{\mu\nu} = \omega_{\mu\nu\lambda}\xi^\lambda$. Note that $\sigma_{\mu\nu}$ can also be written as

$$\sigma_{\mu\nu} = V h_\mu{}^{\mu'} h_\nu{}^{\nu'} \nabla_{\mu'}\xi_{\nu'}. \tag{2.21}$$

The tensor $h_{\mu\nu}$ is, of course, the natural Riemannian metric on N. The covariant derivative D_μ associated with $h_{\mu\nu}$ acting, say, on a covector X_μ living on N, is given by

$$D_\mu X_\nu = h_\mu{}^{\mu'} h_\nu{}^{\nu'} \nabla_{\mu'} X_{\nu'}. \tag{2.22}$$

Denoting by $\mathcal{R}_{\mu\nu\lambda\sigma}$ the curvature associated with D_μ, we find, using Equ. (2.9), that

$$\mathcal{R}_{\mu\nu\lambda\sigma} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} h_\lambda{}^{\lambda'} h_\sigma{}^{\sigma'} R_{\mu'\nu'\lambda'\sigma'} + 2V^{-3}\sigma_{\mu\nu}\sigma_{\lambda\rho} - V^{-3}(\sigma_{\lambda[\mu}\sigma_{\nu]\rho} - \sigma_{\rho[\mu}\sigma_{\nu]\lambda}). \tag{2.23}$$

Since N is 3-dimensional, there holds

$$\sigma_{\mu[\nu}\sigma_{\lambda\rho]} = 0, \tag{2.24}$$

so that

$$\mathcal{R}_{\mu\nu\lambda\rho} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} h_\lambda{}^{\lambda'} h_\rho{}^{\rho'} R_{\mu'\nu'\lambda'\rho'} + 3V^{-3}\sigma_{\mu\nu}\sigma_{\lambda\rho}. \tag{2.25}$$

Thus

$$\mathcal{R}_{\mu\nu} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} R_{\mu'\nu'} - V^{-1} R_{\mu\nu'\lambda\rho'}\xi^{\nu'}\xi^{\rho'} + 3V^{-3}\sigma_{\mu\lambda}\sigma_\nu{}^\lambda. \tag{2.26}$$

Using (2.18), Equ. (2.26) finally leads to

$$\mathcal{R}_{\mu\nu} = h_\mu{}^{\mu'} h_\nu{}^{\nu'} R_{\mu'\nu'} + \frac{1}{2}V^{-1} D_\mu D_\nu V + 2V^{-3}\sigma_{\mu\lambda}\sigma_\nu{}^\lambda - \frac{1}{4}V^{-2}(D_\mu V)(D_\nu V). \tag{2.27}$$

From (2.19) we deduce that

$$D^2 V := h^{\mu\nu} D_\mu D_\nu V = -2R_{\mu\nu}\xi^\mu\xi^\nu + \frac{1}{2}V^{-1}(DV)^2 + 2V^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}. \tag{2.28}$$

We now make the following observation: when $\tau_{\mu\ldots\lambda}$ is an arbitrary tensor on N, there holds

$$h_\nu{}^{\nu'} \ldots h_\lambda{}^{\lambda'} \nabla^\mu \tau_{\mu\nu'\ldots\lambda'} = (-V)^{-1/2} D^\mu[(-V)^{1/2}\tau_{\mu\nu\ldots\lambda}]. \tag{2.29}$$

Applying (2.29) to (2.15) it follows that

$$(-V)^{-1/2} D^\mu[(-V)^{1/2}\sigma_{\mu\nu}] = h_\nu{}^{\nu'} R_{\nu'\mu}\xi^\mu. \tag{2.30}$$

Finally, projecting (2.10) down to N, it follows that

$$D_{[\mu}(V^{-2}\sigma_{\nu\rho]}) = 0. \tag{2.31}$$

Given the spacetime $(M, g_{\mu\nu})$ with the Killing vector ξ^μ, under the conditions stated at the beginning of this section, there are coordinates (t, x^i)

on M, such that the canonical projection π takes the form $\pi : (t, x^i) \mapsto (x^i)$, with x^i local coordinates on N and such that the Killing vector ξ^μ takes the form $\xi = \partial/\partial t$. In terms of such coordinates tensor fields on N, say $\tau_{i\ldots j}(x)$, can be viewed as the tensor fields

$$\tau_{\mu\ldots\nu}(t, x) = \delta_\mu{}^i \ldots \delta_\nu{}^j \tau_{i\ldots j}(x). \tag{2.32}$$

Since $\xi_\mu \xi^\mu = V$, there holds

$$\xi_\mu dx^\mu = V(dt + \varphi_i dx^i), \tag{2.33}$$

for some 1-form φ_i. Note that, in the tangent space at each point $(t, x^i) \in M$, the $g_{\mu\nu}$-orthogonal complement of ξ_μ is spanned by $\varphi_i \partial/\partial t + \partial/\partial x^i$ and the orthogonal complement of ξ_μ in the cotangent space is spanned by dx^i. From the definition $h_{\mu\nu} = g_{\mu\nu} - V^{-1}\xi_\mu \xi_\nu$ it follows that

$$g_{\mu\nu} dx^\mu dx^\nu = V(dt + \varphi_i dx^i)^2 + h_{ij} dx^i dx^j, \tag{2.34}$$

where V, φ_i, h_{ij} on the r.h. side of (2.34) are all independent of t. It is now straightforward to check that

$$\sigma_{\mu\nu} dx^\mu dx^\nu = 3(\xi_{[\mu} \nabla_\nu \xi_{\lambda]}) dx^\mu dx^\nu = V^2 \partial_{[i} \varphi_{j]} dx^i dx^j. \tag{2.35}$$

Thus $\sigma_{\mu\nu}$, viewed as a tensor on N, is given by

$$\sigma_{ij} = V^2 \partial_{[i} \varphi_{j]}. \tag{2.36}$$

In the static case t can be chosen so that $\varphi_i = 0$.

Conversely, let us start from the 3-manifold $(N, h_{ij}, V, \sigma_{ij})$ with Riemannian metric h_{ij}, a negative scalar field V and the 2-form σ_{ij}, subject to

$$D_{[i}(V^{-2}\sigma_{jk]}) = 0, \tag{2.37}$$

which corresponds to (2.31). Suppose, moreover, that N has trivial second cohomology. Then there exists a covector φ_i on N with

$$\sigma_{ij} = V^2 D_{[i} \varphi_{j]}. \tag{2.38}$$

Define $M = \{t \in \mathbf{R}\} \times N$ and define on N the Lorentz metric $g_{\mu\nu}$ by Equ. (2.34) and ξ^μ by $\xi = \partial/\partial t$. Then one checks that $\xi_\mu \xi^\mu = V$, that, under the projection $\pi : M \to N$, $h_{\mu\nu}$ is the pull-back of h_{ij} and that $\sigma_{\mu\nu}$ is the pull-back of $\sigma_{ij} = V^2 D_{[i}\varphi_{j]}$. The fact that the product structure of M as $M = \mathbf{R}^1 \times N$ is not natural is reflected in the above construction by the fact that φ_i, solving (2.38), is given only up to $\varphi_i \mapsto \bar{\varphi}_i = \varphi_i + D_i F$, with F a scalar field on N. Under this change $g_{\mu\nu}$ given Equ. (2.34) remains unchanged only when we set $t \mapsto \bar{t} = t - F$.

Given the fields (h_{ij}, V, σ_{ij}) on N, we can define the fields r, r_i, r_{ij} by the following equations:

$$\begin{aligned}
D^2V &= -2r + \frac{1}{2}V^{-1}(DV)^2 + 2V^{-2}\sigma_{ij}\sigma^{ij} && (2.39)\\
D^i[(-V)^{-1/2}\sigma_{ij}] &= (-V)^{1/2}r_j && (2.40)\\
\mathcal{R}_{ij} &= r_{ij} + \frac{1}{2}V^{-1}D_iD_jV + 2V^{-3}\sigma_{ik}\sigma_j{}^k - \frac{1}{4}V^{-2}(D_iV)(D_jV). && (2.41)
\end{aligned}$$

It then follows from our previous considerations that the spacetime $(M, g_{\mu\nu})$ satisfies

$$R_{\mu\nu}dx^\mu dx^\nu = r(dt + \varphi_\ell dx^\ell)^2 + 2r_i dx^i(dt + \varphi_\ell dx^\ell) + r_{ij}dx^i dx^j. \qquad (2.42)$$

In particular, iff r, r_i, r_{ij} are all zero, $(M, g_{\mu\nu})$ is a vacuum spacetime. In this case we refer to (2.39,40,41) as 'the vacuum equations'.

For later use we record another form of the field equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad (2.43)$$

where

$$T_{\mu\nu}dx^\mu dx^\nu = \tau(dt + \varphi_i dx^i)^2 + 2\tau_i(dt + \varphi_j dx^j)dx^i + \tau_{ij}dx^i dx^j, \qquad (2.44)$$

and where we set

$$g_{\mu\nu}dx^\mu dx^\nu = -e^{2U}(dt + \varphi_i dx^i)^2 + e^{-2U}\bar{h}_{ij}dx^i dx^j, \qquad (2.45)$$

given by

$$\begin{aligned}
\bar{D}^2U &= \frac{\kappa}{2}(e^{-4U}\tau + \bar{\tau}_\ell{}^\ell) - e^{4U}\bar{\omega}_{ij}\bar{\omega}^{ij} && (2.46)\\
\bar{D}^i\bar{\omega}_{ij} &= \kappa e^{-4U}\tau_j && (2.47)\\
\bar{\mathcal{R}}_{ij} &= 2(D_iU)(D_jU) - 2e^{4U}\bar{\omega}_{ik}\bar{\omega}_j{}^k + \bar{h}_{ij}e^{4U}\bar{\omega}_{k\ell}\bar{\omega}^{k\ell} + \kappa(\tau_{ij} - \bar{h}_{ij}\bar{\tau}_\ell{}^\ell). && (2.48)
\end{aligned}$$

Here

$$\bar{\omega}_{ij} = \omega_{ij} = \partial_{[i}\varphi_{j]} \qquad (2.49)$$

and indices are raised with $\bar{h}^{ij}$.

2.2 Axial Symmetry

We now assume the existence of a second, spacelike Killing vector η^μ on $(M, g_{\mu\nu})$. There is the following identity

$$4\nabla^\mu(\eta_{[\rho}\omega_{\mu\nu\lambda]}) = -\mathcal{L}_\eta\omega_{\nu\lambda\rho} + 6\xi^\mu R_{\mu[\nu}\xi_\lambda\eta_{\rho]}, \qquad (2.50)$$

where $\omega_{\mu\nu\lambda}$ is given by Equ. (2.2) and we have used (2.14). Suppose, in addition, that ξ and η commute. Then the first term on the right in (2.50) vanishes so that

$$4\nabla^\mu(\eta_{[\rho}\omega_{\mu\nu\lambda]}) = 6\xi^\mu R_{\mu[\nu}\xi_\lambda\eta_{\rho]}. \tag{2.51}$$

In an analogous manner

$$4\nabla^\mu(\xi_{[\rho}\omega'_{\mu\nu\lambda]}) = -6\eta^\mu R_{\mu[\nu}\xi_\lambda\eta_{\rho]}, \tag{2.52}$$

where $\omega'_{\mu\nu\lambda}$ is given in terms of η in the same way as $\omega_{\mu\nu\lambda}$ is given in terms of ξ. The r.h. sides of Equ.'s (2.51,52) are zero (at points where ξ and η are linearly independent) iff the timelike 2-plane spanned by ξ and η is invariant under $R_\mu{}^\nu$. These conditions will be satisfied when the energy momentum tensor is that of a rotating perfect fluid. We now assume that η^μ has an axis, i.e. vanishes on a timelike 2-surface which is tangent to ξ^μ. Then, and when the r.h. sides of (2.51,52) are zero, it follows that

$$\eta_{[\rho}\omega_{\mu\nu\lambda]} = \xi_{[\rho}\omega'_{\mu\nu\lambda]} = 0. \tag{2.53}$$

The relations (2.53), in turn, are nothing but the conditions for the 2-plane elements orthogonal to ξ and η to be integrable ("surface transitivity of ξ and η"). The above result is due to Kundt and Trümper [38].

For the purposes of Sect. 4.2 we need to transcribe the relations satisfied by η^μ on the quotient manifold N. Writing the 1-form $\eta_\mu = g_{\mu\nu}\eta^\nu$ as

$$\eta_\mu dx^\mu = \eta(dt + \varphi_i dx^i) + \eta_i dx^i, \tag{2.54}$$

so that

$$\eta^\mu \frac{\partial}{\partial x^\mu} = (V^{-1}\eta - \varphi_i\eta^i)\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial x^i}, \tag{2.55}$$

the Killing equations

$$\eta^\lambda\partial_\lambda g_{\mu\nu} + 2g_{\lambda(\mu}\partial_{\nu)}\eta^\lambda = 0 \tag{2.56}$$

are equivalent to

$$\eta^i D_i V = 0 \tag{2.57}$$
$$2\omega_{ij}\eta^j = D_i(V^{-1}\eta) \tag{2.58}$$
$$\mathcal{L}_\eta h_{ij} = 0, \tag{2.59}$$

where $\omega_{ij} := D_{[i}\varphi_{j]}$. The surface transitivity conditions (2.53) get translated into

$$\eta_{[i}D_j\eta_{k]} = 0 \tag{2.60}$$
$$\eta_{[i}\omega_{jk]} = 0. \tag{2.61}$$

In particular, η^i is a hypersurface-orthogonal Killing vector on (N, h_{ij}). Suppose, now, that the energy momentum tensor is that of a rigidly rotating perfect fluid, i.e.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \tag{2.62}$$

with

$$u_\mu = f(\xi_\mu + \Omega\eta_\mu), \qquad \Omega = \text{const}, \tag{2.63}$$

and f is chosen so that u^μ is future-pointing and $u_\mu u^\mu = -1$. With this specialization the quantities τ, τ_i, τ_{ij} entering in the field equation (2.46,47,48) become

$$\tau \ = f^2(\rho + p)(-e^{2U} + \Omega\eta)^2 - pe^{2U} \tag{2.64}$$

$$\tau_i \ = f^2(-e^{2U} + \Omega\eta)(\rho + p)\Omega\eta_i \tag{2.65}$$

$$\tau_{ij} = pe^{-2U}\bar{h}_{ij} + f^2\Omega^2(\rho + p)\eta_i\eta_j, \tag{2.66}$$

where $\eta_i = h_{ij}\eta^j$. The normalization factor f is given by

$$f = [e^{-2U}(-e^{2U} + \Omega\eta)^2 - \Omega^2\eta_\ell\eta^\ell]^{-1/2}. \tag{2.67}$$

The field equations have to be supplemented by the Killing relations (2.57,58,59). Note that these imply that ρ and p are invariant under η^i (in addition of course to being invariant under $\partial/\partial t$). Under these circumstances the contracted Bianchi identities, which imply that

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.68}$$

boil down to the relation

$$(\rho + p)f^{-1}D_i f = D_i p, \tag{2.69}$$

the remaining condition, namely $\bar{D}^j(e^{-4U}\tau_j) = 0$, being identically satisfied.

2.3 Asymptotic Flatness – Lichnerowicz Theorems

Before stating the conditions for stationary spacetimes to be asymptotically flat, we elaborate somewhat more on the vacuum field equations. First recall from (2.17) that, when M (or equivalently: N) is simply connected, there exists a field ω on N such that

$$\sigma_{ij} = \frac{1}{2}(-V)^{1/2}\varepsilon_{ij}{}^k D_k\omega, \tag{2.70}$$

where we have used (2.16) and

$$\varepsilon_{ijk}dx^i dx^j dx^k = (-V)^{1/2}\xi^\mu\varepsilon_{\mu\nu\lambda\sigma}dx^\nu dx^\lambda dx^\sigma. \tag{2.71}$$

(Of course, the existence of ω could have also been inferred from (2.40) for $r_i = 0$.) We now rewrite the vacuum equations in terms of the conformally rescaled metric $\bar{h}_{ij}$ (see (2.45)), given by

$$\bar{h}_{ij} = (-V)h_{ij}. \tag{2.72}$$

Then (2.39,40,41), together with (2.70) lead to

$$\bar{D}^2 V = V^{-1}(\bar{D}V)^2 - V^{-1}(\bar{D}\omega)^2 \tag{2.73}$$

$$\bar{D}^2 \omega = 2V^{-1}(\bar{D}\omega)(\bar{D}V) \tag{2.74}$$

and

$$\bar{\mathcal{R}}_{ij} = \frac{1}{2}V^{-2}[(D_i V)(D_j V) + (D_i\omega)(D_j\omega)], \tag{2.75}$$

or

$$\bar{G}_{ij} = \frac{1}{2}V^{-2}\left\{(D_i V)(D_j V) + (D_i\omega)(D_j\omega) - \frac{1}{2}\bar{h}_{ij}[(\bar{D}V)^2 + (\bar{D}\omega)^2]\right\}. \tag{2.76}$$

We can now give an interesting geometric interpretation of the vacuum equations (2.73,74,76). Namely, let $\mathcal{P}$ be the Poincaré half-plane with metric q_{AB} given by

$$q_{AB}dz^A dz^B = V^{-2}(dV^2 + d\omega^2) \quad (V > 0, -\infty < \omega < \infty). \tag{2.77}$$

Viewing $(z^1(x), z^2(x)) = (V(x), \omega(x))$ as a map from $(N, \bar{h}_{ij})$ to $(\mathcal{P}, q_{AB})$, one easily checks that Equ.'s (2.73,74) are exactly the conditions in order for this map to be harmonic, in other words

$$\bar{D}^2 z^A + \Gamma^A_{BC} z^B{}_{,j} z^C{}_{,j} \bar{h}^{ij} = 0, \tag{2.78}$$

where Γ^A_{BC} denotes the Christoffel symbols of q_{AB}, composed with $z^C(x)$. The metric $\bar{h}_{ij}(x)$, of course, is not given, but has to satisfy (2.51). The r.h. side of Equ. (2.51), in turn, is nothing but the energy momentum tensor of the harmonic map. $(\mathcal{P}, q_{AB})$ can also be viewed as a spacelike hyperboloid in (2+1)-dimensional Minkowski space. Namely, define fields

$$\Phi_M = \frac{V^2 + \omega^2 - 1}{-4V}, \qquad \Phi_S = -\frac{\omega}{2V}, \qquad \Phi_K = -\frac{V^2 + \omega^2 + 1}{4V}. \tag{2.79}$$

Then

$$-\Phi_K^2 + \Phi_M^2 + \Phi_S^2 = -\frac{1}{4}. \tag{2.80}$$

Viewing (Φ_K, Φ_M, Φ_S) as coordinates on $\mathbf{R}^3$ with Lorentz metric $4(-d\Phi_K^2 + d\Phi_M^2 + d\Phi_S^2)$, the induced metric under the map (2.79) is nothing but q_{AB}. The fields Φ_M, Φ_S are the potentials first introduced by Hansen [25] which we shall use in Sect. 3.1 and 3.2.

As with any harmonic map, we can associate a conserved current on N with any Killing vector on the target space $\mathcal{P}$. Since $\mathcal{P}$ has $SO(2,1)$ as isometry group, there are three independent such Killing vectors, namely

$$\overset{1}{\eta}{}^A \frac{\partial}{\partial z^A} = \frac{\partial}{\partial \omega} \tag{2.81}$$

$$\overset{2}{\eta}{}^A \frac{\partial}{\partial z^A} = V\frac{\partial}{\partial V} + \omega \frac{\partial}{\partial \omega} \tag{2.82}$$

$$\overset{3}{\eta}{}^A \frac{\partial}{\partial z^A} = \omega V\frac{\partial}{\partial V} + \frac{1}{2}(\omega^2 - V^2)\frac{\partial}{\partial \omega}. \tag{2.83}$$

We note in passing that the $SO(2,1)$ isometry of $\mathcal{P}$ is closely related to the "Ehlers transformation" discussed in the article by Maison in this volume. The conserved current j_i associated with any Killing vector η^A on $\mathcal{P}$ is given by

$$j_i = z^A{}_{,i}\eta^B q_{AB}. \tag{2.84}$$

Hence

$$\overset{1}{j}_i = V^{-2} D_i \omega \tag{2.85}$$

$$\overset{2}{j}_i = V^{-1} D_i V + V^{-2}\omega D_i \omega \tag{2.86}$$

$$\overset{3}{j}_i = V^{-1}\omega D_i V + (2V)^{-2}(\omega^2 - V^2) D_i \omega \tag{2.87}$$

are all divergence-free on $(N, \bar{h}_{ij})$. By (2.70) and (2.38), $\overset{1}{j}_i$ is also equal to

$$\overset{1}{j}_i = \bar{\varepsilon}_i{}^{jk} D_j \varphi_k, \tag{2.88}$$

and so the "charge" associated with $\overset{1}{j}_i$ is always zero. In the asymptotically flat we shall turn to later, (2.87) will be identically zero.

The quantity (2.86) has the following spacetime interpretation (compare [23]). Let Σ be a 2-surface in M which projects down to a smooth 2-surface on N. Then there exist local coordinates $(x^\mu) = (t, x^i)$ such that Σ is given by

$$x^\mu(y^A) = (0, x^i(y^A)), \qquad A = 1, 2. \tag{2.89}$$

Now integrate the quantity $\varepsilon_{\mu\nu\rho\sigma}\nabla^\rho \xi^\sigma$ over Σ. After some computation one finds

$$\varepsilon_{\mu\nu\rho\sigma}(\nabla^\rho \xi^\sigma)\frac{\partial x^\mu}{\partial y^1}\frac{\partial x^\nu}{\partial y^2} = (-V)^{-1/2}(\partial_i V - 2\sigma_{ij}\varphi^j)\varepsilon^i{}_{k\ell}\frac{\partial x^k}{\partial y^1}\frac{\partial x^\ell}{\partial y^2} \tag{2.90}$$

where, as before, $\sigma_{ij} = V^2 \partial_{[i}\varphi_{j]}$. Now, using (2.70),

$$-2(-V)^{-1/2}\sigma_{ij}\varphi^j + (-V)^{-3/2}\omega D_i \omega = D^j(\omega \varepsilon_{ij}{}^k \varphi_k). \tag{2.91}$$

Thus, when Σ is closed, the integral I of the expression (2.91) is given by ($\bar{h}_{ij} = (-V)h_{ij}$)

$$I = \int_{\Sigma} (-V)^{-1}(D_i V + V^{-1}\omega D_i \omega) d\bar{S}^i. \tag{2.92}$$

The fact that this integral in vacuum only depends on the homology class of Σ arises, in the spacetime picture, from the fact that

$$\nabla^{\mu}\nabla_{[\mu}\xi_{\nu]} = 0, \text{ when } R_{\mu\nu} = 0. \tag{2.93}$$

The quantity

$$M = \frac{1}{8\pi} I \tag{2.94}$$

is called the Komar mass of $(M, g_{\mu\nu})$. For the Schwarzschild solution it coincides with the Schwarzschild mass when the "outward" orientation is chosen for $d\bar{S}^i$.

We now come to the

Boundary Conditions

Recall that we require $(M, g_{\mu\nu})$ to be connected, simply connected and chronological. Let, in addition, M contain a compact subset K and let $M \setminus K$ be an "asymptotically flat end". (The results of this subsection will remain to be true if $M \setminus K$ consists of finitely many asymptotic ends.) This means that $M \setminus K$ should be diffeomorphic to M_R $(R > 0)$ with

$$M_R = \{(x^0, x^i) \in \mathbf{R}^1 \times (\mathbf{R}^3 \setminus B(R))\} \tag{2.95}$$

with $B(R)$ a closed ball of radius R. In terms of this diffeomorphism, the metric $g_{\mu\nu}$ in $M \setminus K$ has to satisfy that there exists a constant $C > 0$ such that (see [3])

$$|g_{\mu\nu}| + |g^{\mu\nu}| + r^{\alpha}|g_{\mu\nu} - \eta_{\mu\nu}| + r^{1+\alpha}|\partial_{\sigma} g_{\mu\nu}| + r^{2+\alpha}|\partial_{\sigma}\partial_{\rho} g_{\mu\nu}| \le C \tag{2.96}$$

$$g_{00} \le -C^{-1}, \qquad g^{00} \le -C^{-1} \tag{2.97}$$

$$\forall\, X^i \in \mathbf{R}^3 \quad g_{ij}X^i X^j \ge C^{-1} \sum (X^i)^2. \tag{2.98}$$

We assume $\alpha > 1/2$. Furthermore we require $R_{\mu\nu}$ to be zero in $M \setminus K$. (This latter condition could be considerably relaxed.) It now follows that the level set $x^0 = 0$ is a spacelike submanifold of $M \setminus K$ which has a finite ADM-momentum p^{μ} (see [3]). If p^{μ} is a timelike vector (which it will be for 'reasonable' matter except in the vacuum case), it now follows from the timelike character of ξ^{μ} that it has to be an asymptotic time translation, i.e.

$$|\xi^{\mu} - A^{\mu}| + r|\partial_{\sigma}\xi^{\mu}| + r^2|\partial_{\rho}\partial_{\sigma}\xi^{\mu}| \le C r^{-\alpha}, \tag{2.99}$$

where the constants A^μ satisfy

$$A^\mu A^\nu \eta_{\mu\nu} < 0 \tag{2.100}$$

(see [3]). Furthermore it follows from [3], that in $M \setminus K$, or a subset thereof diffeomorphic to $M_{R'}$ for sufficiently large $R' > R$, there are coordinates (t, y^i) in terms of which $g_{\mu\nu}$ is again asymptotically flat with the same $\alpha > 1/2$ and so that ξ^μ is of the form $\xi^\mu\ \partial/\partial x^\mu = \partial/\partial t$. Hence, in the coordinates (t, y^i), which we now call (t, x^i), the metric

$$g_{\mu\nu} dx^\mu dx^\nu = V(dt + \varphi_i dx^i)^2 + h_{ij} dx^i dx^j \tag{2.101}$$

satisfies

$$|V+1| + r|\partial_i V| + r^2|\partial_i \partial_j V| \leq Cr^{-\alpha} \tag{2.102}$$
$$|\varphi_i| + r|\partial_j \varphi_i| + r^2|\partial_k \partial_j \varphi_i| \leq Cr^{-\alpha} \tag{2.103}$$
$$|h_{ij} - \delta_{ij}| + r|\partial_k h_{ij}| + r^2|\partial_k \partial_\ell h_{ij}| \leq Cr^{-\alpha} \tag{2.104}$$

in $M \setminus \mathcal{C}$. It follows that

$$r|\sigma_{ij}| + r^2|\partial_k \sigma_{ij}| \leq Cr^{-\alpha}. \tag{2.105}$$

We remark that the time coordinate t, which is at first only defined in the open subset $M \setminus K$ of M, can be (Kobayashi–Nomizu [36]) extended to a smooth global cross section of $\pi : M \to N$.

We now state and prove two uniqueness theorems due to Lichnerowicz [41], which are basic for the theory of stationary solutions.

Staticity theorem: Let $(M, g_{\mu\nu}, \xi^\lambda)$ be asymptotically flat with $\alpha > 1/2$, ξ^μ be an asymptotic time translation. If the matter is non-rotating relative to ξ^μ, i.e. r_i in Equ. (2.40) is zero, then the spacetime is static.

Proof: From (2.40) we have

$$D^i[(-V)^{-1/2}\sigma_{ij}] = 0. \tag{2.106}$$

Contract Equ. (2.106) with φ^j, using $\sigma_{ij} = V^2 D_{[i}\varphi_{j]}$. It follows that

$$D^i[(-V)^{-1/2}\sigma_{ij}\varphi^j] = (-V)^{-5/2}\sigma_{ij}\sigma^{ij}. \tag{2.107}$$

Now integrate Equ. (2.107) over N. Since the term in brackets on the left is $O(r^{-2-2\alpha})$, the boundary term at infinity gives zero. Consequently $\sigma_{ij} = 0 \Rightarrow \omega_{\mu\nu\lambda} = 0$.

Remark: Since $\sigma_{ij} = 0$, the field φ_i is of the form $\varphi_i = D_i F$, where $F = O(r^{1-\alpha})$. In the coordinates $\bar{t} = t - F$, $g_{\mu\nu}$ takes the form

$$g_{\mu\nu} dx^\mu dx^\nu = V dt^2 + h_{ij} dx^i dx^j. \tag{2.108}$$

Vacuum theorem: Let $(M, g_{\mu\nu}, \xi^\lambda)$ satisfy the conditions in the staticity theorem and let $(M, g_{\mu\nu})$ in addition be vacuum. Then $(M, g_{\mu\nu})$ is the Minkowski space.

Proof: Firstly, by the staticity theorem, we have that $\sigma_{ij} = 0$. Using this in Equ. (2.39) for $r = 0$ we have with $v := (-V)^{1/2}$ that

$$D^2 v = 0. \tag{2.109}$$

By the maximum principle, or multiplying (2.109) by μ, integrating by parts and using $\mu - 1 = O(r^{-\alpha})$, $\partial_i \mu = O(r^{-1-\alpha})$, we infer that $\mu \equiv 1$. Now Equ. (2.41) implies $\mathcal{R}_{ij} = 0 \Rightarrow \mathcal{R}_{ijk\ell} = 0$ since $\dim N = 3$. Since N is simply connected, it follows that (N, h_{ij}) is flat $\mathbf{R}^3$. Thus

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \delta_{ij} dx^i dx^j. \tag{2.110}$$

2.4 Newtonian Limit

Ehlers showed (unpublished, see [49]) that one can write the field equation containing a parameter $\lambda = c^{-2}$ such that the equation remain meaningful for $\lambda = 0$ and then they are equivalent to the Newtonian equations. The variables for which this is true in the time dependent case have to be chosen in a quite sophisticated way. The stationary case can be treated in a direct and simple way as follows.

We write the metric as

$$g_{\mu\nu} dx^\mu dx^\nu = -e^{-\frac{2U}{c^2}} (c dt + \varphi_i dx^i)^2 + e^{\frac{2U}{c^2}} \bar{h}_{ik} dx^i dx^k \tag{2.111}$$

where we inserted "c" by dimensional analysis. The field equations decomposed in section 2.1

$$R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) \tag{2.112}$$

for the energy momentum tensor

$$T_{\mu\nu} = c^2 \tau (c dt + \varphi_i dx^i)^2 + 2c\tau_i (c dt + \varphi_j dx^j) dx^i + \tau_{ij} dx^i dx^j \tag{2.113}$$

become

$$\bar{D}^2 U = 4\pi G (e^{-\frac{4U}{c^2}} \tau + c^{-2} \bar{\tau}_l^{\,l}) - e^{\frac{4U}{c^2}} \bar{\omega}_{ij} \bar{\omega}^{ij} \tag{2.114}$$

$$\bar{D}^i \bar{\omega}_{ij} = 8\pi G c^{-3} e^{-\frac{4U}{c^2}} \tau_j \tag{2.115}$$

$$\bar{R}_{ij} = 2c^{-4} D_i U D_j U - 2 e^{\frac{4U}{c^2}} \bar{\omega}_{ik} \bar{\omega}_j{}^k + \bar{h}_{ij} e^{\frac{4U}{c^2}} \bar{\omega}_{kl} \bar{\omega}^{kl} + 8\pi G c^{-4} (\tau_{ij} - \bar{h}_{ij} \bar{\tau}_l^{\,l}) \tag{2.116}$$

Considered as equations for $U, \bar{h}_{ij}, \phi_i, \tau, \tau_i, \tau_{ij}$, the equations (2.114–116) have a limit for $c \to \infty$.

In the static case the limit is

$$\bar{D}^2 U = 4\pi G \tau \tag{2.117}$$

$$\bar{R}_{ik} = 0 \tag{2.118}$$

Hence we obtain immediately that the metric $\bar{h}_{ik}$ of the quotient is flat and therefore (2.117) is the Poisson equation of Newton's theory. The connection has also a limit and the only non vanishing Christoffel symbol is $\Gamma^i_{tt} = \bar{D}^i U$. The equation of motion $\nabla_\nu T^{\mu\nu} = 0$ becomes in the limit the Newtonian equilibrium condition $\bar{D}^j \tau_{ij} = -\tau D_j U$.

Now to the stationary case: Because the right hand side of (2.115) vanishes in the Newtonian limit, the Lichnerowicz theorem implies that $\omega_{ik} = 0$ which in turn implies by (2.116) that $\bar{R}_{ik} = 0$ whence the metric on the quotient is again flat.

For the metric written in the form (1) the connection has no limit for $c \to \infty$. If we use however as a consequence of the field equations that $\omega_{ij} = 0$, the connection has a limit and the equations of motion become the Newtonian equilibrium conditions, i.e.

$$\bar{D}^i \tau_i = 0 \quad , \quad \bar{D}^j \tau_{ij} = -\tau D_j U \ . \tag{2.119}$$

2.5 Existence Issues and the Newtonian Limit

The fact that the equations can be written to contain $\lambda = c^{-2}$ in such a way that they are analytic in λ and are the Newtonian equations for $\lambda = 0$, suggests to use this structure for existence theory. In this section we will make some remarks about the static case. In section 4.2 an existence theorem for a rigidly rotating body by Heilig will be discussed which exploits the fact that the equations have a nice Newtonian limit.

To obtain partial differential equations for which there is an existence theory we write (2.104) and (2.105) in the static case in harmonic coordinates on N, defined by $\bar{D}^2 x^i = 0$, for the unknowns U and Z^{ij} defined by $\bar{h}^{ij} = \delta^{ij} + \lambda^2 Z^{ij}$ and obtain:

$$\Delta U := \delta^{ij}\partial_i\partial_j U = 4\pi G\tau + A(\lambda, \tau, \tau_{ij}, Z^{ij}) \tag{2.120}$$

$$\begin{aligned} \Delta Z^{ij} &= -4\partial_k U \partial_l U \delta^{ik}\delta^{lj} - 16\pi G(\bar{\tau}^{ij} - \bar{\tau}\delta^{ij}) \\ &\quad + \lambda^2 B^{ij}(\lambda, \tau, \tau_{kl}, Z^{kl}, \partial_m Z^{kl}, \partial_m\partial_n Z^{kl}) \end{aligned} \tag{2.121}$$

here we used the well known expression for the Ricci tensor in harmonic coordinates

$$\bar{R}^{ij} = -\frac{1}{2}\bar{h}^{kl}\partial_k\partial_l\bar{h}^{ij} + H^{ij}(\partial\bar{h}, \partial\bar{h}) \tag{2.122}$$

where H^{ij} is quadratic in the first derivatives of $\bar{h}^{ij}$. As usual we call (2.120), (2.121) the reduced field equations. These form a quasilinear elliptic system with the property that for given small sources τ, τ_{ij} of compact support and small λ there exist unique solutions U, Z^{ij} which tend to 0 at infinity.

In particular we can choose for τ, τ_{ij} a Newtonian solution and determine then for small λ a relativistic solution of the reduced field equations which have a Newtonian limit. Is is to be expected that the solution will be analytic in λ. Then the Taylor expansion in λ can be considered as a converging post Newtonian expansion.

A solution of the reduced field equations is only solution of the field equation if it satisfies the harmonicity condition or equivalently if $\nabla_\mu T^{\mu\nu} = 0$ holds. To solve the reduced equations and the equation of motion is a much harder problem. It makes only sense once a matter model is chosen.

In the static case only matter models of elasticity lead to new interesting problems because, as we will see in sections 5.2 , static fluids are spherically symmetric and can be investigated by ordinary differential equations (see Sect. 5.3). Some remarks on static, small self gravitating bodies can be found in Sect. 5.1.

For a stationary rigidly rotating fluid Heilig has given an existence theorem by perturbing away from a Newtonian solution. We will describe this result in Sect. 4.2.

3 Far Fields

3.1 Far-Field Expansions

While, as we have seen, little is known so far about globally regular, asymptotically flat solutions to the stationary field equations with reasonable matter sources, there is an almost complete understanding of the behaviour of general asymptotically flat solutions near spatial infinity, which we now describe.

Here the quotient manifold N is of the form

$$N = \mathbf{R}^3 \setminus B(R).$$

On N there are given the fields (h_{ij}, V, φ_i) satisfying (2.39,40,41). The whole discussion is "local-at-infinity". In particular one has to allow for R in $B(R)$ to be made suitably large, as one proceeds. We will do so tacitly without changing the letter "R". The Einstein equations are given by

$$\bar{D}^2 V = V^{-1}(\bar{D}V)^2 - V^{-1}(\bar{D}\omega)^2 \tag{3.1}$$

$$\bar{D}^2 \omega = 2V^{-1}(\bar{D}\omega)(\bar{D}V) \tag{3.2}$$

$$\bar{G}_{ij} = \frac{1}{2}V^{-2}\{(D_i V)(D_j V) + (D_i\omega)(D_j\omega) - \frac{1}{2}\bar{h}_{ij}[(\bar{D}V)^2 + (\bar{D}\omega)^2]\} \tag{3.3}$$

By the asymptotic conditions (2.102–104), ω tends to a constant at infinity. Subtracting this from ω, and calling the result again ω, we find that

$$|\omega| + r|\partial_i\omega| + r^2|\partial_i\partial_j\omega| \le Cr^{-\alpha}. \tag{3.4}$$

In short, we have that

$$V = -1 + O(r^{-\alpha}), \qquad \omega = O(r^{-\alpha}), \qquad \bar{h}_{ij} - \delta_{ij} = O(r^{-\alpha}), \qquad 1 > \alpha > 1/2 \tag{3.5}$$

and that these relations may be differentiated twice. The condition $\alpha > 1/2$ could be relaxed (see Kennefick and Ó Murchadha [34]). It now follows that

$$\Delta V = O(r^{-2-2\alpha}) \tag{3.6}$$

$$\Delta\omega = O(r^{-2-2\alpha}). \tag{3.7}$$

Since the r.h. sides of (3.6,7) decay stronger than $O(r^{-3})$, it follows from standard results in potential theory [33,73] that there exist constants M, S such that

$$V = -1 + \frac{2M}{r} + O(r^{-1-\alpha}) \tag{3.8}$$

$$\omega = \frac{2S}{r} + O(r^{-1-\alpha}). \tag{3.9}$$

But the existence of φ_i in Equ. (2.36) implies that S has to be zero.

The equation (3.3), which involves second derivatives of the metric $\bar{h}_{ij}$, yields

$$\Delta(k_{ij} - \frac{1}{2}\delta_{ij}k) - 2\partial_{(i}\Gamma_{j)} + \delta_{ij}\partial_\ell\Gamma_\ell = O(r^{-2-2\alpha}), \tag{3.10}$$

where $k_{ij} = \bar{h}_{ij} - \delta_{ij}$, $k = k_{ii}$ and

$$\Gamma_i = \partial_j k_{ij} - \frac{1}{2}\partial_i k. \tag{3.11}$$

This equation can be viewed in two ways, both of which recur in the higher-order steps leading to the theorem below. Firstly, in the gauge where $\Gamma_i = 0$, i.e. the linearized harmonic gauge for $\bar{h}_{ij}$, it is an elliptic equation, namely essentially the componentwise Laplace equation, for the leading-order part of k_{ij}. Secondly, Equ. (3.10) can be rewritten as

$$\varepsilon_{i\ell m}\varepsilon_{jnp}\partial_\ell\partial_n k_{mp} = O(r^{-2-2\alpha}), \tag{3.12}$$

which expresses the fact that the linearized Riemann tensor of $\bar{h}_{ij}$ decays faster than $O(r^{-3})$. Note that (3.12) makes essential use of the three-dimensionality of space. It follows [73] that there exists $g_i = O(r^{1-\alpha})$ such that

$$k_{ij} = \partial_i g_j + \partial_j g_i + O(r^{-2\alpha}). \tag{3.13}$$

Thus the leading-order contribution to the metric $\bar{h}_{ij}$ is "pure gauge".

To next order in $1/r$ one finds that there is a gauge, namely

$$\Gamma_i = O(r^{-3-\alpha}), \tag{3.14}$$

for which

$$V = -1 + \frac{2M}{r} - \frac{2M_i x^i}{r^3} + \frac{2M^2}{r^2} + O(r^{-1-2\alpha}) \tag{3.15}$$

$$\omega = \frac{2S_i x^i}{r^3} + O(r^{-1-2\alpha}) \tag{3.16}$$

and for which $\bar{h}_{ij}$ can be brought into the form

$$k_{ij} = -\frac{M^2(\delta_{ij} r^2 - x_i x_j)}{r^4} + O(r^{-1-2\alpha}). \tag{3.17}$$

In the above M, M_i, S_i are constants. All indices are lowered and raised with δ_{ij}. When $M \neq 0$ one can, by a rigid translation, arrange for $M_i = 0$. In that case the metric $g_{\mu\nu}$ obtained from (3.15–17) coincides, to order $1/r^2$, with that of the Kerr spacetime with $|S| = -Ma$, M being the mass and a being the Kerr parameter.

In order to extend the above result to higher orders in $1/r$, it is convenient to replace (V, ω) by some other choice of scalar potentials. One choice, due to Hansen [25], is to set (see Equ. (2.79)

$$\phi_M = -\frac{V^2 + \omega^2 - 1}{4V} \tag{3.18}$$

$$\phi_S = -\frac{\omega}{2V} \tag{3.19}$$

$$\phi_K = -\frac{V^2 + \omega^2 + 1}{4V} \tag{3.20}$$

It then turns out that $(\phi_\alpha) = (\phi_M, \phi_S, \phi_K)$ all satisfy

$$\bar{D}^2 \phi_\alpha = 2\bar{\mathcal{R}} \phi_\alpha. \tag{3.21}$$

Then one has [73] the following

Theorem: There exists a gauge, namely that where $\Gamma_i = O(r^{-m-1-\alpha})$, for which there are constants $A\ldots, B\ldots, \ldots, G\ldots$ such that

$$\phi_M = \sum_{\ell=0}^{m-1} \frac{E_{i_1\ldots i_\ell} x^{i_1}\ldots x^{i_\ell}}{\ell! r^{2\ell+1}} + O^\infty(r^{-m+1-2\alpha}) \tag{3.22}$$

$$\phi_S = \sum_{\ell=0}^{m-1} \frac{F_{i_1\ldots i_\ell} x^{i_1}\ldots x^{i_\ell}}{\ell! r^{2\ell+1}} + O^\infty(r^{-m+1-2\alpha}) \tag{3.23}$$

$$\phi_K = \frac{1}{2} + \sum_{\ell=0}^{m-1} \frac{G_{i_1\ldots i_{\ell-1}} x^{i_1}\ldots x^{i_{\ell-1}}}{\ell! r^{2\ell}} + O^\infty(r^{-m+1-2\alpha}) \tag{3.24}$$

Note that $E = M$,

$$\bar{h}_{ij} = \delta_{ij} + \sum_{\ell=2}^{m} \left(\frac{x_i x_j A_{i_1\ldots i_{\ell-2}} x^{i_1}\ldots x^{i_{\ell-2}}}{r^{2\ell}} + \frac{\delta_{ij} B_{i_1\ldots i_{\ell-2}} x^{i_1}\ldots x^{i_{\ell-2}}}{r^{2\ell-2}} \right.$$
$$\left. + \frac{x_{(i} C_{j) i_1\ldots i_{\ell-3}} x^{i_1}\ldots x^{i_{\ell-3}}}{r^{2\ell-2}} + \frac{D_{ij i_1\ldots i_{\ell-4}} x^{i_1}\ldots x^{i_{\ell-4}}}{r^{2\ell-4}} + O^\infty(r^{-m+1-2\alpha}) \right). \tag{3.25}$$

All constants are symmetric in their $i_1 \ldots$ indices. The constants D are also symmetric in i and j. The constants $C \ldots$ appear only for $m \geq 3$, the constants $D \ldots$ only for $m \geq 4$. The symbol $O^\infty(r^k)$ means that the quantity in question is of $O(r^k)$, its derivative is $O(r^{k-1})$, a.s.o. Furthermore all constants are determined by the tracefree parts of $E \ldots, F \ldots$ in a way which does not depend on the solution at hand. The tracefree parts of $E \ldots$ are the analogues of the Newtonian multipole moments. The constants $F \ldots$ play an analogous role for the "angular-momentum aspect", which does not have a Newtonian counterpart. The three-metric $\bar{h}_{ij}$, for reasons explained after (3.14), has no independent degrees of freedom.

This theorem shows, in essence, that any stationary, asymptotically flat solution to the Einstein vacuum equations is uniquely determined by the "moments" $E \ldots, F \ldots$. However no statement concerning convergence of series like the ones appearing in (3.22–25) can be made. In order to do that it is necessary to use "conformal compactification" of three-space N.

3.2 Conformal Treatment of Infinity, Multipole Moments

Before turning to the situation G.R., it is instructive to recall the Newtonian situation. Suppose we are given a Newtonian potential near infinity, i.e. a function ϕ with

$$\Delta\phi = 0 \qquad \text{on } \mathbf{R}^3 \setminus B(R). \tag{3.26}$$

Extending ϕ smoothly to all of $\mathbf{R}^3$, we thus have that

$$\Delta\phi = 4\pi\rho \qquad \text{with } \rho \in C_0^\infty(\mathbf{R}^3) \tag{3.27}$$

and $\phi \to 0$ at infinity. Thus ϕ is of the form

$$\phi(x) = -\int_{\mathbf{R}^3} \frac{\rho(x')}{|x - x'|} dx'. \tag{3.28}$$

In $\mathbf{R}^3 \setminus B(R)$ this can (see e.g. [33]) be expanded in a standard fashion in powers of $1/r$. One obtains an expansion of the form

$$\phi = \sum_{\ell=0}^{\infty} \frac{E_{i_1 \ldots i_\ell} x^{i_1} \ldots x^{i_\ell}}{\ell! r^{2\ell+1}}, \tag{3.29}$$

with $E_{i_1 \ldots i_\ell}$ totally symmetric and tracefree. One shows [33,73] that this series converges absolutely and uniformly in $\mathbf{R}^3 \setminus B(R)$ for sufficiently large R.

As a warm-up for G.R. it is useful to rephrase the Newtonian situation using "conformal compactification". First observe that there is a positive smooth function Ω on $N = \mathbf{R}^3 \setminus B(R)$ with the following properties. The metric

$$\widetilde{h}_{ij} = \Omega^2 \delta_{ij} \tag{3.30}$$

extends to a smooth metric on the one-point compactification

$$\widetilde{N} = N \cup \{r = \infty\} = N \cup \{\Lambda\}, \tag{3.31}$$

where

$$\Omega|_\Lambda = 0, \qquad \widetilde{D}_i \Omega|_\Lambda = 0 \tag{3.32}$$

and

$$\widetilde{D}_i \widetilde{D}_j \Omega - 2\widetilde{h}_{ij} = 0. \tag{3.33}$$

To prove this, take $\Omega = 1/r^2$ and introduce

$$\widetilde{x}^i = \frac{x^i}{r^2} \tag{3.34}$$

as coordinates on $\widetilde{N}$. One also sees that $\widetilde{h}_{ij}$ is again the standard flat metric in the coordinates $\widetilde{x}^i$. (This would also follow from (3.33) and the standard formula for the behaviour of R_{ij} under conformal rescalings.) As for the potential, rewrite (3.26) as

$$\left(D^2 - \frac{\mathcal{R}}{6} \right) \phi = 0, \tag{3.35}$$

and observe that the operator in (3.35) obeys

$$\left(\widetilde{D}^2 - \frac{\widetilde{\mathcal{R}}}{6} \right) \widetilde{\phi} = \Omega^{-5/2} \left(D^2 - \frac{\mathcal{R}}{6} \right) \phi, \tag{3.36}$$

when $\widetilde{h}_{ij} = \Omega^2 h_{ij}$ and $\widetilde{\phi} = \Omega^{-1/2}\phi$ for arbitrary $\Omega > 0$. Thus we again have

$$\left(\widetilde{D}^2 - \frac{\widetilde{\mathcal{R}}}{6} \right) \widetilde{\phi} = \widetilde{D}^2 \widetilde{\phi} = 0, \tag{3.37}$$

at first only on N.

In the case of G.R. we were unable to prove convergence of the multipole series, but only an asymptotic estimate like

$$\phi = \sum_{\ell=0}^{m-1} \frac{E_{i_1 \dots i_\ell} x^{i_1} \dots x^{i_\ell}}{\ell! \, r^{2\ell+1}} + O^\infty(r^{-m+1-2\alpha}). \tag{3.38}$$

But, from (3.38) for $m = 4$, it follows immediately that $\widetilde{\phi}$ extends to a C^3-function on $\widetilde{N}$. Thus, by continuity

$$\widetilde{D}^2 \widetilde{\phi} = 0 \qquad \text{on } \widetilde{N}. \tag{3.39}$$

But it is a standard fact that solutions to the Laplace equation and, more generally, for nonlinear elliptic systems with analytic coefficients [55], are themselves analytic. Thus $\widetilde{\phi}$ has a convergent Taylor expansion at the point Λ. But this is nothing but (3.29) in inverted coordinates. Furthermore the multipole moments $E_{i_1 \dots i_\ell}$ can now be viewed as the Taylor coefficients of $\widetilde{\phi}$ at Λ. It follows from (3.39) that they have to be tracefree, and it is trivial that they determine $\widetilde{\phi}$ uniquely.

Suppose Ω is just required to satisfy (3.32,33). Then, given h_{ij}, there is in $(\widetilde{h}_{ij}, \Omega)$ the following 3-parameter gauge freedom

$$\Omega' = \omega \Omega, \tag{3.40}$$

$$\widetilde{h}'_{ij} = \omega^2 \widetilde{h}_{ij}, \tag{3.41}$$

where

$$\omega = (1 - b^i \widetilde{D}_i \Omega + b^i b_i \Omega)^{-1}, \tag{3.42}$$

with $\widetilde{D}_i b^j = 0$, which, in the compactified picture, corresponds to the freedom of choosing an origin in the "physical" space $\mathbf{R}^3$, w.r.t. to which the inversion $\widetilde{x}^i = x^i / r^2$ can be made. Therefore the Taylor coefficients of $\widetilde{U}$ at Λ behave under (3.40,41) in a way which precisely corresponds to their dependence on the choice of origin.

In G.R. it is impossible to require a conformal compactification for which (3.33) holds everywhere. We call a 3-metric $\bar{h}_{ij}$ on a manifold $N \cong \mathbf{R}^3 \setminus B(R)$ conformally C^k, when there exists a C^k-function $\Omega > 0$ on N such that $\widetilde{h}_{ij} = \Omega^2 \bar{h}_{ij}$ extends to a C^k-metric on $\widetilde{N} = N \cup \{\Lambda\}$ and

$$\Omega|_\Lambda = 0, \qquad \widetilde{D}_i \Omega|_\Lambda = 0, \tag{3.43}$$

$$(\widetilde{D}_i \widetilde{D}_j \Omega - 2\widetilde{h}_{ij})\Big|_\Lambda = 0. \tag{3.44}$$

A scalar potential ϕ is called conformally C^k, when $\widetilde{\phi} = \Omega^{-1/2}\phi$ extends to a C^k-function on $\widetilde{N}$. Given (3.43,44) there is now a much larger gauge freedom involved in constructing the unphysical from the physical quantities, namely

$$\Omega' = \omega \Omega, \qquad \widetilde{h}'_{ij} = \omega^2 \widetilde{h}_{ij}, \qquad \widetilde{\phi}' = \omega^{-1/2}\phi \tag{3.45}$$

where ω satisfies $\omega|_\Lambda = 1$. Now consider, following Geroch [22], this recursively defined set of tensor fields on $\widetilde{N}$:

$$P_0 = \widetilde{\phi} \tag{3.46}$$

$$P_i = D_i \widetilde{\phi} \tag{3.47}$$

$$P_{ij} = TS\left[\widetilde{D}_i D_j \widetilde{\phi} - \frac{1}{2}\widetilde{\mathcal{R}}_{ij}\widetilde{\phi}\right] \tag{3.48}$$

$$P_{i_1 \dots i_{m+1}} = TS\left[\widetilde{D}_{i_{m+1}} P_{i_1 \dots i_m} - \frac{s(2s-1)}{2}\widetilde{\mathcal{R}}_{i_1 i_2} P_{i_3 \dots i_{m+1}}\right], \tag{3.49}$$

where TS denotes the operation of taking the symmetric, trace-free part. It turns out that the tensors

$$E_{i_1\dots i_m} = P_{i_1\dots i_m}|_\Lambda \tag{3.50}$$

behave under (3.45) in exactly the same way as the Newtonian moments under the restricted gauge freedom (3.40–42) with $b_i = \widetilde{D}_i\omega|_\Lambda$. Thus the Ricci terms in (3.46–49) cancel out unwanted dependencies from higher-than-first derivatives of ω at Λ.

Now return to the expansions (3.22–25) for some fixed $m \geq 1$. Performing, again, an inversion $\widetilde{x}^i = x^i/r^2$ and setting, in these coordinates,

$$\widetilde{\phi}_M = \Omega^{-1/2}\phi_M, \qquad \widetilde{\phi}_S = \Omega^{-1/2}\phi_S, \tag{3.51}$$

$$\widetilde{h}_{ij} = \Omega^2 \bar{h}_{ij} \tag{3.52}$$

with $\Omega = 1/r^2$ we find that $(\widetilde{\phi}_M, \widetilde{\phi}_S, \widetilde{h}_{ij})$ are all C^m. Furthermore Ω is C^∞. Thus we have obtained a C^m conformal compactification. Our proof would be complete if we could find an elliptic system satisfied by $(\widetilde{h}_{ij}, \widetilde{\phi}_M, \widetilde{\phi}_S)$ or quantities derived from them. Doing this is not completely trivial. We explain the essentials in the static case where $\phi_S = 0$. Thus

$$\bar{D}^2\phi_M = 2\bar{\mathcal{R}}\phi_M \tag{3.53}$$

$$\bar{\mathcal{R}}_{ij} = \frac{2}{1+4\phi_M^2}(D_i\phi_M)(D_j\phi_M). \tag{3.54}$$

Let us assume that $M \neq 0$. Define, instead of $1/r^2$ as above, a conformal factor also called Ω by

$$\Omega = \frac{[(-V)^{1/2}-1]^2}{(-V)^{1/2}}. \tag{3.55}$$

It is not hard to see from (3.22–25) that this yields a C^m-compactification $(\widetilde{\phi}_M, \widetilde{h}_{ij})$ where, however, we have for convenience replaced (3.44) by

$$\left.\left(\widetilde{D}_i\widetilde{D}_j\Omega - \frac{2}{M^2}\widetilde{h}_{ij}\right)\right|_\Lambda = 0. \tag{3.56}$$

It is useful to employ, as the scalar variable in the unphysical picture neither $\widetilde{\phi}_M$ nor Ω, but the quantity σ defined by

$$\sigma := \left[\frac{(-V)^{1/2}-1}{(-V)^{1/2}+1}\right]^2. \tag{3.57}$$

After some labor we find from (3.53,54) that

$$\widetilde{\mathcal{R}} = 0 \tag{3.58}$$

and

$$-\sigma(1-\sigma)\widetilde{\mathcal{R}}_{ij} = \widetilde{D}_i\widetilde{D}_j\sigma - \frac{1}{3}\widetilde{h}_{ij}\widetilde{D}^2\sigma. \tag{3.59}$$

The scalar σ satisfies

$$\sigma|_\Lambda = 0, \qquad \widetilde{D}_i\sigma|_\Lambda = 0, \qquad \widetilde{D}^2\sigma|_\Lambda = \frac{3}{2M^2}. \tag{3.60}$$

Taking a "curl" of Equ. (3.59) we obtain

$$(1-\sigma)\widetilde{D}_{[i}\widetilde{\mathcal{R}}_{j]k} = 2(\widetilde{D}_{[i}\sigma)\widetilde{R}_{j]k} - \widetilde{h}_{k[i}\widetilde{R}_{j]\ell}\widetilde{D}^\ell\sigma. \tag{3.61}$$

If we take $\widetilde{D}^i$ of the quantity $\widetilde{D}_{[i}\widetilde{\mathcal{R}}_{j]k}$ and use the Ricci and Bianchi identities we find the relation

$$\widetilde{D}^2\widetilde{\mathcal{R}}_{jk} = \frac{1}{2}\widetilde{D}_j\widetilde{D}_k\widetilde{\mathcal{R}} + 2\widetilde{D}^i\widetilde{D}_{[i}\widetilde{\mathcal{R}}_{j]k} + 3(\widetilde{\mathcal{R}}_{ji}\widetilde{\mathcal{R}}^i{}_k - \frac{1}{2}\widetilde{\mathcal{R}}\widetilde{\mathcal{R}}_{jk}) - \frac{1}{2}\widetilde{h}_{jk}(\widetilde{\mathcal{R}}_{i\ell}\widetilde{\mathcal{R}}^{i\ell} - \frac{1}{2}\widetilde{\mathcal{R}}^2). \tag{3.62}$$

Using that $\widetilde{\mathcal{R}}$ is zero and Equ. (3.62), writing $\widetilde{\mathcal{R}}_{ij} = \tau_{ij}$, and using (3.59) to eliminate second derivatives of σ, we obtain an equation of the form

$$\widetilde{D}^2\tau_{ij} = \text{nonlinear terms}, \tag{3.63}$$

where these nonlinear terms depend at most on τ_{ij}, σ and their first derivatives and on $\widetilde{D}^2\sigma$. We call $\widetilde{D}^2\sigma = \rho$. From the divergence of Equ. (3.59) we infer that

$$\rho\sigma = \frac{3}{2}(\widetilde{D}\sigma)^2, \tag{3.64}$$

and from this after some work that

$$\widetilde{D}^2\rho = 3\sigma(1-\sigma)^2\widetilde{\mathcal{R}}_{ij}\widetilde{\mathcal{R}}^{ij} + 3\widetilde{\mathcal{R}}_{ij}(\widetilde{D}^i\sigma)(\widetilde{D}^j\sigma). \tag{3.65}$$

Now Equ. (3.63) can be completed as follows:

$$\widetilde{\mathcal{R}}_{ij} = \tau_{ij} \tag{3.66}$$
$$\widetilde{D}^2\tau_{ij} = \text{nonlinear terms} \tag{3.67}$$
$$\widetilde{D}^2\sigma = \rho \tag{3.68}$$
$$\widetilde{D}^2\rho = 3\sigma(1-\sigma)^2\widetilde{\mathcal{R}}_{ij}\widetilde{\mathcal{R}}^{ij} + 3\widetilde{\mathcal{R}}_{ij}(\widetilde{D}^i\sigma)(\widetilde{D}^j\sigma). \tag{3.69}$$

Going over to harmonic coordinates, the "non-elliptic" terms in the expression of $\widetilde{\mathcal{R}}_{ij}$ in (3.66) in terms of the metric go away, and the whole set of Equ.'s (3.66–69) becomes an elliptic system. Note that the point of the whole manœuvre was that the original Equ. (3.59), when written in terms of $\widetilde{h}_{ij}$ is singular since $\sigma|_\Lambda = 0$. The miracle was that, in the transition from (3.59) to (3.61) a factor σ is obtained on both sides of (3.61) which can be cancelled since σ is nonzero outside Λ by (3.57).

Thus, taking m sufficiently large and appealing to the theorem of Morrey [55], we have the

Theorem: When $M \neq 0$, there is a chart in a neighbourhood of Λ for which $(\sigma, \widetilde{h}_{ij})$ are analytic. Consequently, from (3.57), Ω is also analytic, and so is $\widetilde{\phi}_M = (1-\sigma)^{-3/2}$.

An analogous result can be proved for a suitable set $(\widetilde{h}_{ij}, \Omega, \widetilde{\phi}_M, \widetilde{\phi}_S)$ in the stationary case [6], see also [39]. The equations one obtains imply in particular that the "physical" quantities $(\bar{h}_{ij}, \phi_M, \phi_S)$ have an analytic chart in a neighbourhood of each point of N and thus entail the "classic" result of Müller zum Hagen on the analyticity of stationary vacuum solutions [57].

By smoothness of $(\widetilde{h}_{ij}, \widetilde{\phi}_M, \widetilde{\phi}_S)$ we can define multipole moments for each of $\widetilde{\phi}_M, \widetilde{\phi}_S$, following (3.46–50). One can show [73] that they coincide with the quantities $E\dots$ and $F\dots$ in the expansions (3.22–25). (These, in turn, coincide with the ones in Thorne [74], as shown in [24]). One can now prove [6], that these moments determine the stationary solution uniquely up to isometries. We give a more careful formulation of this result only in the static case.

Theorem: Let there be two static solutions with the same $\widetilde{h}_{ij}|_\Lambda$, the same $M \neq 0$ and the same set of (mass-centered) multipole moments. Then the corresponding physical solutions $(\bar{h}_{ij}, \phi_M)$ are isometric.

The proof is a not-too-difficult inductive argument based on (3.61,62), (3.68,69) and (3.59,60).

There remains the question to what degree the multipole moments of stationary solutions can be prescribed. It is fairly easy to see, e.g. from the asymptotic analysis of Sect. 3.1, that the multipole moments are "algebraically independent", i.e. for a given finite number of them, there always exists a spacetime having those moments which solves the stationary field equations to arbitrary order in $1/r$. It is not known what conditions on moments for high order have to be imposed in order for the multipole expansion to converge. In particular, convergence is not even known when only finitely many moments are non-zero.

There are of course solution-generating techniques to in principle write down the general stationary axisymmetric spacetime. To date the only result on existence of stationary asymptotically flat solutions without any further symmetry is that of Reula [66].

We note, in passing that the above equations lend themselves to an easy proof of a result which is often used in black-hole uniqueness theorems (see [31]). Namely an asymptotically flat, static vacuum solution with $M \neq 0$, which is spatially conformally flat, has to be isometric to the Schwarzschild metric near Λ. To see this, use that now the Cotton tensor of $\widetilde{h}_{ij}$ is zero. Thus, since $\widetilde{\mathcal{R}} = 0$, the left hand side of Equ. (3.61) vanishes. Contracting the r.h. side of (3.61) with $(\widetilde{D}^i\sigma)\widetilde{\mathcal{R}}^{jk}$ we find that

$$2(\widetilde{\mathcal{R}}_{ij}\widetilde{\mathcal{R}}^{ij})(\widetilde{D}\sigma)^2 = (\widetilde{\mathcal{R}}_{ij}\widetilde{D}^j\sigma)(\widetilde{\mathcal{R}}^i{}_\ell \widetilde{D}^\ell\sigma). \tag{3.70}$$

But, by Cauchy–Schwarz, the right-hand side of (3.70) is bounded above by

$$(\widetilde{\mathcal{R}}_{ij}\widetilde{\mathcal{R}}^{ij})(\widetilde{D}\sigma)^2,$$

which has hence to be zero. Since σ can not have critical points near Λ except at Λ itself, it follows that

$$\widetilde{\mathcal{R}}_{ij} = 0, \tag{3.71}$$

whence, from (3.59), $\widetilde{D}^2\sigma = 3/2M^2$ and thus, in a chart $\widetilde{x}^i$ for which $\widetilde{h}_{ij} = \delta_{ij}$ we have $\sigma = |\widetilde{x}|^2/4M^2$, from which it easily follows that $(\bar{h}_{ij}, \phi_M)$ corresponds to Schwarzschild with mass M.

4 Global Rotating Solutions

4.1 Lindblom's Theorem

Lindblom showed in his thesis [45] that stationary asymptotically flat dissipative fluid configuration are axisymmetric. In this section we want to outline and discuss this theorem.

There are three ingrediences of the proof:

(i) The local fluid field equations imply that the fluid flow is proportional to a Killing vector t^μ provided the divergence of the entropy current vanishes.
(ii) The Killing field t^μ has an extension into the vacuum field of the solution.
(iii) If the manifold of orbits of the stationary Killing vector ξ^μ is R^3 and asymptotically flat, then ξ^μ is linearly independent of t^μ. The two Killing fields commute and there is a linear combination of the two Kiliing fields which has fixed points near which it act like a rotation.

(i) Theorem: Let $g_{\mu\nu}$, $T_{\mu\nu}$ be a stationary local solutions of the Einstein field equations for a one–component fluid with phenomenological heat conduction and viscosity laws and vanishing of the divergence of the entropy current. Then the fluid flow is proportional to a Killing vector.

Proof: The energy momentum tensor for a fluid with shear and bulk viscosity is [54] (θ and $\sigma_{\mu\nu}$ are the expansion and shear of the fluid; q^μ is the heat flow [20])

$$T^{\mu\nu} = \rho u^\mu u^\nu + (p - \zeta\theta)h^{\mu\nu} - 2\eta\sigma^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu \tag{4.1}$$

with

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \ , \ q_\mu u^\mu = 0, \quad \sigma_{\mu\nu} u^\mu = 0 \ . \tag{4.2}$$

This implies (a dot denotes the covariant derivative in the direction of the fluid flow u^μ)

$$0 = -(\nabla_\mu T^{\mu\nu})u_\nu = \dot{\rho} + (\rho + p)\theta - \zeta\theta^2 - 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \nabla_\mu q^\mu + q_\mu \dot{u}^\mu \ . \tag{4.3}$$

Introducing n, the conserved rest–mass density, and the specific volume $v = \frac{1}{n}$ and the specific internal energy $u = \frac{\rho}{n}$ we can rewrite this, using $\nabla_\mu(nu^\mu) = 0$, as

$$n(\dot{u} + p\dot{v}) - \zeta\theta^2 - 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \nabla_\mu q^\mu + q_\mu \dot{u}^\mu = 0 \ . \tag{4.4}$$

For a one–component fluid we have an equation of state $u = u(p, v)$ and consequently there exist scalar functions $T(p, v)$ and $s(p, v)$ with the interpretation of temperature and specific entropy such that

$$du + pdv = Tds \ . \tag{4.5}$$

Hence $n(\dot{u} + p\dot{v}) = nT\dot{s}$ can be used to rewrite (4.4) as

$$nT\dot{s} - \zeta\theta^2 - 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \nabla_\mu q^\mu + q_\mu \dot{u}^\mu = 0 \tag{4.6}$$

or

$$n\dot{s} + T^{-1}\nabla_\mu q^\mu = T^{-1}(\zeta\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} - q_\mu \dot{u}^\mu) = 0 \ . \tag{4.7}$$

Using again $\nabla_\mu(nu^\mu) = 0$ we obtain

$$\nabla_\mu(nsu^\mu + T^{-1}q^\mu) = T^{-1}[\zeta\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} - q_\mu(\dot{u}^\mu + T^{-1}\nabla^\mu T)] = 0 \ . \tag{4.8}$$

Inserting the phenomenological law of heat conduction

$$q_\mu = -\kappa h^\nu{}_\mu (T_{,\nu} + T\dot{u}_\nu) \tag{4.9}$$

we obtain

$$\nabla_\mu(nsu^\mu + T^{-1}q^\mu) = T^{-1}(\zeta\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu} + \kappa T^{-1}q_\mu q^\mu) = 0 \ . \tag{4.10}$$

The left hand side of this equation is the conserved entropy current $\nabla_\mu s^\mu$ which vanishes according to our assumptions. Hence the positivity of λ, ζ and κ implies $\theta = \sigma^{\mu\nu} = q^\mu = 0$ and $\dot{u}_\mu = -T^{-1}T_{,\mu}$.

Assume $T \neq 0$ and consider $\xi^\mu = T^{-1}u^\mu$, the candidate for the Killing vector. We have

$$\nabla_{(\mu}\xi_{\nu)} = -T^{-2}\nabla_{(\mu}Tu_{\nu)} + T^{-1}\nabla_{(\mu}u_{\nu)} \ . \tag{4.11}$$

The vanishing of $\theta = \sigma_{\mu\nu} = q^\mu = 0$ implies $\nabla_{(\mu}u_{\nu)} = -\dot{u}_{(\mu}u_{\nu)} = T^{-1}(\nabla_{(\mu}T)u_{\nu)}$, hence $\nabla_{(\mu}\xi_{\nu)} = 0$.

Now we come to the most complicated part, the extension of the Killing vector proportional to the fluid flow from the fluid into the surrounding vacuum region.

(ii) Conjecture: Let $g_{\mu\nu}, T_{\mu\nu}$ be a strictly stationary, asymptotically flat perfect fluid solution where the matter is a ball of finite extent and the fluid flow is proportional to a Killing vector t^μ. Then t^μ has a unique extension into the vacuum region, provided certain differentiability conditions are satisfied at the boundary.

In Lindblom's original treatment this conjecture was shown to be true under the assuption that the outside metric is analytic up to and including the boundary Σ of the fluid. Then one can propagate the Killing vector into a neighbourhood of the boundary using the Cauchy Kowalevskaja theorem because a Killing vector satisfies a wave type equation. Finally a theorem by Nomizu [61] can be used to obtain a global Killing vector field.

One might, however argue, that analyticity up to and including Σ is too strong an assumption. On physical grounds one would like to treat also non–analytic equations of state. In this case it is unlikely that the metric is analytic in the boundary.

Finally we show that the new Killing vector t^μ is actually different from the stationary Killing vector ξ^μ.

(iii) Theorem: Under the assumption of the above conjecture we have:

(1) The Killing vectors ξ^μ and t^μ are linearly independent.

(2) Both Killing vectors commute.

(3) There exists a linear combination $\eta^\mu = t^\mu + a\xi^\mu$ which has fixed points and acts like a rotation with closed orbits.

Proof: (1) Suppose t^μ would be linearly dependent of ξ^μ. Then there would be a timelike Killing vector, namely t^μ, which is asymptotically a translation and relative to which the matter does not rotate. Hence, by the Licherowicz staticity theorem, spacetime would have to be static.

(2) As T and u^μ are invariant objects we have $\mathcal{L}_\xi T = 0$ and $\mathcal{L}_\xi u^\nu = 0$ which imply immediately $[\xi, t] = 0$ on the support of the matter. To show that this is also true outside the matter one can use the analyticity of the outer metric up to and including the boundary or one can use a theorem by Beig and Chrusciel [4] classifying all possible group action on asymptotically flat spacetimes.

(3) As ξ^μ commutes with t^μ there is a Killing vector $\hat{t}^i$ on the manifold of orbits of ξ^μ. The corresponding group acts in the 2–surface of constant pressure, in particular in the boundary, $p = 0$. As this is topologically S^2, there must be a point where $\hat{t}^i$ vanishes. A Killing vector on a Riemannian space with a fixed point acts always as a rotation with closed circular orbits. At a point q in spacetime projecting on the fixed point of $\hat{t}^i$, t^μ must be proportional to ξ^μ and therefore a linear combination $\eta^\mu = t^\mu + a\xi^\mu$ with constant coefficients vanishing at q exist such that $\eta^\mu(q) = 0$. We have a fixed point and because also the timelike direction of ξ^μ is fixed, η^μ acts like a rotation and has therefore closed orbits.

We see that we can obtain the existence of the axis working only on the body, provided we know that both Killing vectors are independent. Lindblom [43] obtains the axis and commutativity of the Killing vectors from the asymptotic symmetry group. The key property that the two Killing vectors are linearly independent is only implied by a global argument and uses asymptotic flatness.

4.2 Existence of Stationary Rotating Axi-Symmetric Fluid Bodies

Following work by Liapunoff and Poincaré, Lichtenstein [42] demonstrated at the beginning of this century the existence of rotating fluid bodies in Newtonian theory. An account of this almost forgotten work can be found in [71]. Using implicit function theorem techniques — as we would say today — he shows the existence of solutions near known solutions or approximately known solutions: starting with a static fluid ball, he obtains a slowly rotating fluid ball; starting from a self gravitating 2–body point particle solution, he obtains a solution for two small fluid bodies orbiting their center of mass on a circle. Furtheremore, there is a number of exact solutions in Newtonian theory: the Maclaurin spheroids, the Jacobi and the Dedekind ellipsoids and the Riemann ellipsoids [18].

In Einstein's theory we do not know any stationary exact solution describing an extended rotating body. Spacetimes describing such solutions can be characterized as follows: Besides a timelike Killing vector ξ^μ there is a further symmetry, the axial symmetry generated by η^μ, whose orbits are circles (Remember that we showed in Sect. 4.1 that such an extra symmetry exists on physical grounds) The body is spatially compact and the spacetime with topology R^4 is assumed to be asymptotically flat. We assume that there is an axis where η^μ vanishes. Then we can use a result of Carter [16] which states that under these circumstances the two Killing vectors commute. Such spacetimes are called "stationary axisymmetric", the orbits of the axial Killing vector are circles.

We showed in Sect 2.2 that for stationary axisymmetric perfect fluids with an axis and a fluid flow vector contained in the two-surface spanned by the two Killing vectors, the two-surface elements orthogonal to the two-dimensional group orbit are surface forming (the group action is orthogonally transitive). The same holds in the vacuum region. The property of orthogonal transitivity is equivalent to the existence of a discrete isotropy group [70].

To introduce a global coordinate system let us assume that outside the 2-dimensional axis the spacetime is the product of the orbits of the isometry group and the orthogonal 2 – surface which we assume to have topology R^2.

Using coordinate adapted to the Killing vectors the metric can be written as

$$ds^2 = g_{AB}(x^c)dx^A dx^B + g_{00}(x^C)dt^2 + 2g_{0\phi}(x^C)dt d\phi + g_{\phi\phi}(x^C)d\phi^2. \quad (4.12)$$

Locally we can always introduce coordinates $(x^A) = (r, z)$ such that g_{AB} is conformal to the flat metric in standard coordinates and can therefore write the metric as

$$ds^2 = e^{2k-2U}(dr^2 + dz^2) + e^{-2U}W^2 d\phi^2 - e^{2U}(dt + Ad\phi)^2. \quad (4.13)$$

There is the freedom in (r, z) of an arbitrary conformal transformation which is given by the real part of analytic function.

The function W^2 is the volume element of the group orbits. As a consequence of the field equations in vacuum one can locally achieve $W = r'$ such that

$$ds^2 = e^{2k'-2U}(dr'^2 + dz'^2) + e^{-2U}r'^2 d\phi^2 - e^{2U}(dt + Ad\phi)^2. \qquad (4.14)$$

These coordinates are called Weyl's canonical coordinates. Matters can be arranged so that $r' = 0$ is the axis. Then the coordinates are fixed up to a translation in z'.

It is tempting to try to extend the Weyl coordinates from the outside of the body to the interior such that the two-surface orthogonal to the group orbit is covered by one (r', z') system with $r' = 0$ describing the axis and $W \neq r'$ in the interior. However, Müller zum Hagen has demonstrated [56] that this is impossible in the case of static spherically symmetric solutions. (r' becomes negative inside the body and the axis is reached for $\rho' \to \infty$.) There is no reason to assume that this would be different in the stationary case.

Numerical codes work successfully with a global (r, z) coordinate system such that $r = 0$ is the axis but it is not assumed that one has Weyl's canonical coordinate in vacuum.

For perfect fluids whose velocity is proportional to a constant linear combination of the two Killing vectors, the case of rigid rotation, $\nabla_\nu T^{\mu\nu} = 0$ becomes particularly simple. (See equation (2.69).)

$$0 = \nabla_\nu T_\mu{}^\nu = (\rho + p)\frac{1}{2}(\ln f^{-2})_{,\mu} + p_{,\mu}. \qquad (4.15)$$

where $f^\mu = f^2(\xi^\mu + \Omega\eta^\mu)$ is the four velocity of the fluid. This shows that, provided an equation of state $\rho(p)$ is given, the matter variables p and ρ can be expressed as functions of the quantity f which is determined by the geometry. This property of rigidly rotating fluids is essential for all the numerical schemes as well as for all the attempts to prove existence.

Various authors have developed codes to calculate numerically stationary, axisymmetric rigidly rotating fluid solutions [12]. Today this can be done with very high presicion by different numerical techniques. These numerical solutions are also the basis for investigations of oscillations of rotating stars.

Schaudt and Pfister [68] try to obtain an existence theorem working in the above coordinates adapted to the symmetry. This approach is attractive because the field equations become semilinear elliptic. One has, however, to control the singularities in the equations on the axis. This is possible and two Dirichlet problems have been solved, which give existence of outside, asymptotically flat solutions and existence of inner parts of bodies, provided appropriate boundary values are given [67]. Up to now this was only possible for the "outer" and the "inner" problem separately and work is in progress which tries to combine the inner and outer solution.

Let us now turn to the discussion of the only existence theorem for rotating fluids in Einstein's theory. It is remarkable that the first existence theorem for rotating fluids, proved by Heilig in 1995 [30], uses Lichtenstein's technique and does not adapt the coordinate to the axial Killing vector to avoid difficulties at the axis.

Let us formulate one particular case of the theorem proved by Heilig [30].

Theorem: Let $\rho(p) = Cp^\gamma$ be a polytropic equation of state with $1 < \gamma < 6/5$. The central density ρ_0 determines a unique Newtonian static fluid ball solution of finite extent. Then there exist a positive constant Ω_0 such that for all Ω with $0 < \Omega < \Omega_0$ a stationary axisymmetric rigidly rotating solution with angular velocity Ω of the Einstein field equations for a perfect fluid exists. The solution is geodesically complete, asymptotically flat with finite mass and angular momentum. The matter is of finite extent and has the same equation of state and central density as the Newtonian solution.

The theorem holds also for more general equations of state. It is not clear whether the case of positive boundary density may be treated by this method.

Heilig uses the observation of Jürgen Ehlers [21] that it is possible to write the field equations as an elliptic system with a parameter $\lambda = c^{-2}$ — interpreted as the velocity of light — such that the equations for , $\lambda \to 0$ give the Newtonian equations and the limit is regular. This can be achieved by a particular choice of unknowns for which the field equations are formulated.

We will describe the structure of Heiligs proof using the equations formulated in Sect. 2.4 because these are much simpler. We want, however, to stress that we expect that Heilig's result could be proved more easily using these equations, but this is not certain before all the functional analysis has been done properly.

Let us first adapt the field equations to a rigidly rotating fluid. We write the axial Killing vector as in (2.54)

$$\eta_\mu dx^\mu = \eta(cdt + \phi_i dx^i) + \eta_i dx^i \ . \tag{4.16}$$

The Killing equation in spacetime is equivalent to the equations (2.57–59) on the quotient N. For a rigidly rotating perfect fluid with fluid flow vector u^μ we have

$$u_\mu = f(\xi_\mu + \Omega\eta_\mu), \quad \Omega = \text{const} \ , \quad u_\mu u^\mu = -c^2, \tag{4.17}$$

where

$$f^{-2} = e^{-\frac{2U}{c^2}}(-e^{\frac{2U}{c^2}} + c^{-1}\Omega\eta)^2 - c^{-2}\Omega^2\eta_l\eta^l \ . \tag{4.18}$$

To obtain the field equation we replace in (2.64–66) η by $c^{-1}\eta$ and p by $c^{-2}p$ to obtain from (2.46–48) using $U \to c^{-2}U$

$$\begin{aligned}
\bar{D}^2 U \ &= 4\pi G[f^2(-e^{\frac{2U}{c^2}} + c^{-1}\Omega\eta)^2(\rho + c^{-2}p) + 2c^{-2}pe^{-\frac{2U}{c^2}}] \\
&\quad + c^{-2}e^{\frac{4U}{c^2}} f^2\Omega^2(\rho + c^{-2}p)\eta_i\eta_j\bar{h}^{ij}]e^{\frac{4U}{c^2}} - e^{\frac{4U}{c^2}}\bar{\omega}_{ij}\bar{\omega}^{ij} && (4.19)\\
\bar{D}^i\bar{\omega}_{ij} &= 8\pi G c^{-3}e^{-\frac{4U}{c^2}} f^2(-e^{\frac{2U}{c^2}} + c^{-1}\Omega\eta)(\rho + c^{-2}p)\Omega\eta_j && (4.20)\\
\bar{R}_{ij} \ &= 2c^{-4}D_iUD_jU - 2e^{\frac{4U}{c^2}}\bar{\omega}_{ik}\bar{\omega}_j{}^k + \bar{h}_{ij}e^{\frac{4U}{c^2}}\bar{\omega}_{kl}\bar{\omega}^{kl}
\end{aligned}$$

$$+ 8\pi G c^{-4}[-2pe^{-\frac{2U}{c^2}}\bar{h}_{ij} + f^2\Omega^2(\rho + c^{-2}p)\eta_i\eta_j - \bar{h}_{ij}f^2\Omega^2(\rho + c^{-2}p)\eta_l\eta_m\bar{h}^{lm}] \tag{4.21}$$

The above field equations have to be supplemented by the Killing equations (2.57–2.59). The equations of motion are

$$\nabla_\mu T^{\mu\nu} = 0 \Longleftrightarrow (c^2\rho + p)f^{-1}D_i f = D_i p \tag{4.22}$$

For $c \to \infty$ we have from (4.18) that $f^2 = 1$ which implies by (4.20) that $\bar{D}^i\omega_{ij} = 0$. The staticity theorem now gives that $\bar{\omega}_{ij} = 0$. Using (2.58) this implies $D_i(e^{-\frac{2U}{c^2}}\eta) = 0$. The vanishing of η on the axis implies $\eta = 0$. Using all this the field equations reduce to

$$\bar{R}_{ij} = 0\ , \quad \bar{D}^2 U = 4\pi G\rho \tag{4.23}$$

Therefore the metric on N is flat. Using $lim_{c\to\infty}(c^2 D_i f) = -D_i(U - \frac{1}{2}\Omega^2\eta_l\eta^l)$ the equation of motion become the Newtonian equation

$$-\rho D_i(U - \frac{1}{2}\Omega^2\eta_l\eta^l) = D_i p \tag{4.24}$$

equation ($\eta_l\eta^l = x^2 + y^2$ in Cartesian coordinates).

As discussed in Sect. 2.5 for the static case, the field equations become again a quasilinear elliptic system for U, Z^{ij}, φ_i in harmonic coordinates ($\nabla_\mu\nabla^\mu t = 0 \Longleftrightarrow \bar{D}^i\varphi_i = 0$, $\nabla_\mu\nabla^\mu x^i = 0 \Longleftrightarrow \bar{D}^2 x^i = 0$). Namely, the condition that the time function is harmonic turns the left-hand side of Equ.(4.20) into an elliptic operator acting on φ_i. Harmonicity of x^i has the same effect on the left hand side of Equ.(4.21). Theorem 4.1 of Heilig [30] can be adapted to show that for small λ, Ω a solution of the reduced field equation exists near the Newtonian solution. Such a solution satisfies only the harmonicity condition if the equation of motion holds. So, this has to be solved simultaneously. This is possible because given a equation of state (4.22) can be integrated such that the matter quantities can be expressed in terms of the geometrical quantity f. Therefore the following iteration procedure is well defined: begin with a Newtonian solution U^0, p^0; choose some λ, Ω and use ρ^0, p^0, U^0 as a source in the field equations in harmonic coordinates to obtain $U^1, Z^{1ij}, \varphi^1{}_i$. Calculate f from $U^1, Z^{1ij}, \varphi^1{}_i, \lambda, \Omega$ and determine p^1 from the equation of motion. Then one solves again the field equation with the new source and so on. Heilig has shown that for sufficiently small λ and Ω such an iteration converges in his variables. It should also converge in the variables used here.

It is remarkable that we have used $\eta_i dx^i = xdy - ydx$ as a given field. At the end one has to check that the solution is axisymmetric and satisfies the harmonicity condition.

Note that only for $\lambda = c^{-2}$ with some fixed value of the velocity of light in some units the above field equations are Einstein's equations. It is however possible to reinterpret solutions with any λ as solutions of Einstein's equation expressed in different units [30]. With this interpretation the theorem above demonstates the existence of slowly — the theorem does not control the range of ω — rotating fluid configurations.

4.3 The Neugebauer–Meinel Disk

The only known global solution describing a rotating object in Einstein's theory, is the relativistic analogue of the rigidly rotating Maclaurin disk in Newton's theory [10].

An axisymmetric surface density distribution (in cylindrical coordinates (r, ϕ, z))

$$\sigma(r) = \sigma_0 \sqrt{1 - \frac{r^2}{r_0^2}} \ , \quad 0 < r < a \ , \tag{4.25}$$

generates a gravitational potential $\Phi(r, z)$, which is determined by the Poisson integral from σ. At the disk the potential is

$$\Phi(r, 0) = \frac{1}{2} \Omega^2 r^2 + \mathrm{const} \ , \quad 0 < \mathrm{r} < \mathrm{a} \ , \quad \Omega^2 = \frac{\pi^2 \mathrm{G} \sigma_0^2}{\mathrm{r}_0^2} \ . \tag{4.26}$$

Outside the disk the potential can be expressed, for example, in terms of integrals over Bessel functions.

The centifugal force acting on rigidly rotating particles balances the gravitational force in the disk. Therefore, we can interpret the density distribution as formed by self gravitating, rigidly rotating dust. The two parameters σ_0 and r_0 determine such disks uniquely. The total mass of the disk is $M = \frac{2}{3}\pi\sigma_0 r_0^2$.

Neugebauer and Meinel found the relativistic analog of these disks [59].

There is a well known formalism available in General Relativity to describe matter surface distributions [31]. In the particular case of a reflection symmetric disk, we have to find solutions of the stationary vacuum field equations, defined outside the disk such that the difference of the normal derivatives of the metric at of the disk have a certain algebraic structure [31].

The general stationary axisymmetric metric can be parametrized as

$$ds^2 = e^{-2U} \left[e^{2k}(dr^2 + dz^2) + r^2 d\phi^2 \right] - e^{2U}(dt + a d\phi)^2. \tag{4.27}$$

The metric coefficients U, k and a depend only on r, z; the vector fields $\xi^\mu \partial/\partial x^\mu = \partial/\partial t$ and $\eta^\mu \partial/\partial x^\mu = \partial/\partial\phi$ are Killing vector fields. We assume that the orbits of the axial Killing vector are circles; $r = 0$ is the axis.

Let us assume that the disk is located at $z = 0, \ 0 \leq r < r_0$. A rigidly rotating flow forming the disk is described by a vector field (which is defined at the disk)

$$u^\mu = e^{-V}(\xi^\mu + \Omega\eta^\mu) \ , \quad u^\mu u_\mu = -1 \ , \tag{4.28}$$

where Ω is constant. The definition of a dust disk implies that the metric is continuous across the disk and that $\tau^{\mu\nu} = \sigma u^\mu u^\nu$, where σ is the surface density, satisfies $\tau^{\mu\nu}{}_{;\nu} = 0$ with respect to the Levi Civita connection of the metric induced on the disk. As $v^\mu = \xi^\mu + \Omega\eta^\mu$ is a Killing vector, it holds $v^\mu{}_{;\mu} = 0$, $\sigma_{,\mu}v^\mu = 0$, $V_{,\mu}v^\mu = 0$ and we obtain

$$\tau^{\mu\nu}{}_{;\nu} = \left(\sigma e^{-2V} v^\mu v^\nu\right)_{;\nu} = \sigma e^{-2V} v^\mu{}_{;\nu} v^\nu . \tag{4.29}$$

Finally $e^{2V} = g_{\mu\nu}v^\mu v^\nu$ implies $e^{2V}2V_{,\gamma} = 2g_{\mu\nu}v^\mu v^\nu{}_{;\gamma} = 2v^\nu v_{\gamma;\nu}$ and we see that V must be constant on a disk formed of rigidly rotating dust, $V = V_0$.

It is natural to introduce comoving coordinates

$$t' = t; \;\; \phi' = \phi - \Omega t \;, \;\; \xi^{\mu'} = \xi^\mu + \Omega\eta^\mu \;, \;\; \eta^{\mu'} = \eta^\mu \;, \;\; u^{\mu'} = e^{-V}\delta^{\mu'}_{t'} . \tag{4.30}$$

The vacuum field equations can be expressed in terms of the following quantities:

$$e^{2U'} = -\xi_{\mu'}\xi^{\mu'} = e^{2V} \;, \;\; a' = -e^{-2U'}\eta_{\mu'}\xi^{\mu'} \;, \; U'(r,\phi,z=0) = V_0 = const \tag{4.31}$$

and $b'(r,z)$ determined by

$$a'_{,r} = re^{-4U'}b'_{,z} \;, \;\; a'_{,z} = -re^{-4U'}b'_{,r} \; . \tag{4.32}$$

Using the Ernst potential $f' = e^{2U'} + ib'$ the key field equation is the semilinear elliptic Ernst equation [37]

$$Re(f')(f'_{,rr} + f'_{,zz} + \frac{1}{r}f'_{,r}) = f'^2_{,r} + f'^2_{,z} \; . \tag{4.33}$$

For a solution of the Ernst equation the integrability condition of (4.32) is satisfied and one can solve for a'. The remaining metric coefficient k' follows from the equations

$$k'_{,r} = r\left[U'^2_{,r} - U'^2_{,z} + \frac{1}{4}e^{-4U'}(b'^2_{,r} - b'^2_{,z})\right] \;, \;\; k'_{,z} = 2r\left[U'_{,r}U'_{,z} + \frac{1}{4}e^{-4U'}(b'_{,r}b'_{,z})\right] \;, \tag{4.34}$$

whose integrability condition is again satisfied for solutions of the Ernst equations.

We can perform an integral in the $(r-z)$ –plane around the disk of the integrability condition of (4.32), namely

$$(r^{-1}e^{4U'}a'_{,r})_{,r} + (r^{-1}e^{4U'}a'_{,z})_{,z} = 0 \;, \tag{4.35}$$

which can be replaced by a surface integral. As we assume that the tangential derivatives of the metric are continuous at the disk, we obtain at the disk

$$a'_{,z}|_{z=0^+} = a'_{,z}|_{z=0^-} \; . \tag{4.36}$$

On the other hand reflection symmetry at the disk implies

$$a'_{,z}|_{z=0^+} = -a'_{,z}|_{z=0^-} \tag{4.37}$$

on the disk, hence,

$$a'_{,z}|_{z=0^+} = a'_{,z}|_{z=0^-} = 0 \ , \tag{4.38}$$

which by (4.32) implies $b' = const$ on the disk.

Now it is easy to calculate the second fundamental form $k_{cd} = \frac{1}{2}e^U g_{cd,z}$ of the disk $z = 0$ ($c, d, \ldots = (t, r, \phi)$), because we have at the disk that $a'_{,z} = k'_{,z} = 0$, as a consequence of (4.38),(4.35) and (4.34). We find

$$k_{rr} = -2U'_{,z} g_{rr} \tag{4.39}$$

$$k_{\phi'\phi'} = -2U'_{,z}(a'^2 e^{2U'} + e^{-2U'} r^2) \tag{4.40}$$

$$k_{t't'} = 2U'_{,z} g_{t't'} \tag{4.41}$$

$$k_{t'\phi'} = 2U'_{,z} g_{t'\phi'} . \tag{4.42}$$

Now we can check the condition for a disk of dust [31], namely

$$^+k_{cd} - {}^-k_{cd} = 2^+k_{cd} = -8\pi(\tau_{cd} - \frac{1}{2}g_{cd}\tau^e_e) = -8\pi(\sigma u_c u_d + \frac{1}{2}\sigma g_{cd}) \ , \tag{4.43}$$

which, in the primed coordinates (because of $u^{\mu'} = \delta^{\mu'}_{t'}$), reads

$$k_{c'd'} = -8\pi\sigma(g_{c't'}g_{d't'} + g_{c'd'}) \ . \tag{4.44}$$

Because of the form of the metric (4.27) in primed coordinates and by (4.39–42), this is satisfied if we define the surface density by

$$\sigma = \frac{1}{2\pi}U'_{,z} \ . \tag{4.45}$$

Thus we have shown that a rigidly rotating disk of dust is determined by a solution of the Ernst equation which satisfied at the disk $U' = const$ and $b' = const$. Outside the disk the solutions of the Ernst equation must be regular. For a well-posed elliptic problem we need furthermore asymptotic flatness at infinity and regularity conditions at the axis.

In Newton's theory there is a 2–parameter family of disks (4.25), (4.26). If we use the 2–parameter group of similarity transformations — or dimensionless quantities — we can assume $r_0 = 1$ and $\sigma_0 = 1$ and we have just one disk.

Because of the appearence of the velocity of light there is only a 1–parameter group of similarity transformations in Einstein's theory. Hence, after we put $r_0 = 1$, we expect a 1–parameter family of disk solutions.

The investigations of Neugebauer and Meinel suggest that

$$\mu = 2\omega^2 r_0^2 e^{-2V_0} \tag{4.46}$$

is an appropriate parameter.

Neugebauer and Meinel prove by the so called inverse scattering method of soliton physics (compare the contribution of Maison in this volume) that the boundary value problem for f' has a unique global solution provided V_0 and ω are such that $\mu < \mu_{crit} = 4.629\ldots$.

The solution f' can be expressed in terms of hyperelliptic theta functions [60]. The remaining metric coefficients a' and k' are determined by integration from the equations (4.32) and (4.34).

If we put the velocity of light, c, in the appropriate places we obtain the MacLaurin disk as a Newtonian limit.

Further properties of these disks are discussed in [60].

Many global stationary solutions with disk sources may be constructed from known stationary vacuum solutions by "cutting out" a region containing singularities an making appropriate identifications. This method was first used by Bicak and Ledvinka [9] to produce physically plausible sources for the Kerr metric with arbitrary values of the parameters a, M. These disks are made of two streams of particles circulating in opposite directions with differential velocities. They are extending to infinity but have finite mass. See Sect. 6 of the article by Bicak in this volume, where this procedure is related back to the "method of images" in Newtonian galactic dynamics. In the static case these methods yield an infinite number of such static disk solutions. Solutions corresponding to stationary counterrotating dust disks of finite extent have been constructed by Klein and Richter [35].

5 Global Nonrotating Solutions

5.1 Elastic Static Bodies

No doubt, Einstein's theory should allow for the description of static, solid bodies. It is useful to make the following distinction:

(i) small bodies, whose shape is not dominated by gravitational forces, like a piece of sugar or an iron ball. If we ignore gravity, the structure of the body is determined by the laws of quantum mechanics. This is true in a Galilei invariant formulation as well as in a special relativistic one. Linear and nonlinear elasticity theory describes the deformation of such a configuration under external forces.

Suppose we now want to add the gravitational field. This is straightforward for linear elasticity in Newtonian theory; we just have to insert the gravitational field calculated from the Poisson integral as an external force into the equations of elasticity.

To pass from special relativity to Einstein's theory is more complicated. Now the deformed configuration has to satisfy Einstein's field equations, and the elasticity equations are a consequence of the latter!

(ii) bodies like stars whose shape is dominated by gravitational forces. There a relaxed state does not really exist and one has to modify the desciption of elasticity. This holds in Newtonian theory as well as in Einstein' theory.

Elastostatics can be described in Einstein's theory as follows [17]. The collection of particles which form the body is described by the three-dimensional "body manifold" B. The essential dynamical variable is a map $\Phi : M^4 \to B$. such that $\Phi^{-1}(y^i)$ is the world line of the particle in spacetime labeled by y^i in B. In the static case the world lines of the particles are the integral curves of the Killing vector, and we can consider Φ as a 1–1 map $N \to B$. We assume that we have given on B a Riemannian metric $\bar{\kappa}_{ij}$. Its physical interpretation may be different: for small bodies it describes a relaxed state; for big bodies which go never into a relaxed state, it could be an "isotropic state of minimal energy".

We need now information about the energy momentum tensor of the material. Let

$$T^{\mu\nu} = \rho u^\mu u^\nu + p^{\mu\nu} \ , \quad p^{\mu\nu} u_\nu = 0 \tag{5.1}$$

be such that the stress tensor $p^{\mu\nu}$ has only spatial components and can be considered as a tensor on N. We can now define

$$e_{ij} := \frac{1}{2}(h_{ij} - \Phi_* \bar{\kappa}_{ij}) \tag{5.2}$$

the "Lagrangian strain tensor". In the Hookian approximation of elastcity one assumes that one has given on the body B a tensor field $\bar{K}^{ijk\ell}$ such that after moving this object with Φ into the space N' one can define

$$\rho = \rho_0 + \frac{1}{2} K^{ijk\ell} e_{ij} e_{k\ell} \tag{5.3}$$

$$p^{ij} = -K^{ijk\ell} e_{k\ell} \tag{5.4}$$

as the energy and stresses of the body B in 3–space with the metric h_{ij}.

With this energy momentum tensor we consider Einstein's field equations as differential equations for the spacetime metric and the map Φ. No general existence theorem is available for this problem. The only case treated so far is the spherically symmetic one [62].

To get some feeling for these equation let us consider some further idealisation. For small deformations we can linearize $e_{ij} := \frac{1}{2}(h_{ij} - \Phi_* \bar{\kappa}_{ij})$ as follows: Suppose that Ψ_ϵ is a 1–parameter family of diffeomorphism $N' \to N'$ such that $\epsilon = 0$ is the identity and Φ_0 is some diffeomorphism $N' \to B$. Now we assume that $\Psi_\epsilon \Phi^{-1}{}_0$ defines our deformed body and calculate the stress tensor e_{ij} to first order in ϵ If we define $\kappa^0_{ij} = \Phi_{0*} \bar{\kappa}_{ij}$ we obtain

$$e_{ij} = \frac{1}{2}(h_{ij} - \kappa^0_{ij} + \mathcal{L}_\chi \kappa^0_{ij}) = \frac{1}{2}(h_{ij} - \kappa^0_{ij} + D^0_{(i} \chi^k \kappa^0_{j)k}) \tag{5.5}$$

Here the vector field χ^i is defined by the linearization of Ψ_ϵ on N' . We see that $p^{ij}{}_{;j} = 0$ leads to second order differential equations for χ^a.

Consider first the case of special relativity which coincides with Galilei invariant classical mechanics in the static situation. Then we have $h_{\alpha\beta} = \eta_{\alpha\beta}$

and $h_{ij} = \delta_{ij}$. We choose Φ_0 to be the identity may which discribes the relaxed body in spacetime. With $\bar{\kappa}_{ij} = \delta_{ij}$ we obtain

$$e_{ij} = \frac{1}{2}\chi_{(i,j)} \tag{5.6}$$

This gives implies the equations of classical, linearized elastostatics [51].

$$0 = p^{ij}{}_{,j} = -K^{ijk\ell}\chi_{k,\ell j} \tag{5.7}$$

With the appropriate symmetry and positivity conditions on $K^{ijk\ell}$ the equations are elliptic and solutions exist for various boundary conditions.

Next we want to calculate the deformation of a small elastic body by its own gravitational field. The relaxed state is determined by solid state physics as above. To switch on gravity we assume that we have families $g_{\mu\nu} = \eta_{\mu\nu} + Gg^1_{\mu\nu} + G^2 g^2_{\mu\nu} \ldots$ and $T_{\mu\nu} = T^0_{\mu\nu} + GT^1_{\mu\nu} + G^2 T^2_{\mu\nu} \ldots$ satisfying the field equations.

At order G^0 we obtain the trivial solution if there are no forces at the body , the density ρ^0_0 is constant and the stresses vanish,i.e. $\chi^0_a = 0$. The field equation in order G are obtained from the equations in section 2.4 with an energy momentum tensor $T^0_{\mu\nu}$ which has only a term ρ^0_0 because the stresses vanish. We obtain U^1 as a solution of the Poisson equation with the source source ρ^0_0. The metric $\bar{h}^1_{ij}$ remains flat in this order. The expansion of the equation of motion in G gives to first order

$$p^{1ij}{}_{,j} = -K^{ijk\ell}\chi^1_{k,\ell j} = -\rho^0_0 U^1{}_{,a} \ , \qquad \Delta U^1 = 4\pi G\rho^0_0 \tag{5.8}$$

Hence we obtain classical elastostatics with the force deforming the body being the gravitational force.

One might try to obtain an existence theorem for small self gravitating elastic bodies in Einstein's theory by an implicit function theorem argument similarly as in the case of a rigidly rotating body (section 4.2).

5.2 Are Perfect Fluids $O(3)$-Symmetric?

It is intuitively "obvious" that a static, in particular non-rotating, ball of perfect fluid, due to the absence of shear stresses should have spherical symmetry, and in particular the gravitational field in its exterior should be the one described by the Schwarzschild spacetime. This result, in its most general form, is still open in G.R. (The Newtonian case was settled in Lichtenstein [42] and Carleman [15], see also Lindblom [44].) Rather, one has today a theorem which is essentially a uniqueness result in the spirit of black hole uniqueness theorems. An earlier result due to Künzle and Savage [40] states that, near a spherical solution, there is no aspherical one with the same equation of state and the same mass.

The following result, due to Beig and Simon [8], is a refinement of previous work by Masood-ul-Alam [52], see also the review of Lindblom [47].

Theorem : Let us have a static, asymptotically flat, spherically symmetric solution to the Einstein equations with a perfect fluid and barotropic equation of state $\rho = \rho(p)$. (This solution is called reference spherical solution.) Let there be given another static, asymptotically flat solution with the same equation of state and the same value $V|_{\partial S}$ of the Killing vector norm on the surface ∂S of the star. Let further $\rho(p)$ satisfy the differential inequality $I \leq 0$, specified later. Then these two spacetimes are isometric, in particular the second one is also $O(3)$-symmetric.

The condition stipulating the existence of a spherical reference solution was disposed of by Lindblom and Masood-ul-Alam [48]. The condition on the matter, besides the one stating that $\rho \geq 0$, $p \geq 0$ and $d\rho/dp \geq 0$, is that

$$I := \frac{1}{5}\kappa^2 + 2\kappa + (\rho + p)\frac{d\kappa}{dp} \leq 0, \tag{5.9}$$

where $\kappa := \frac{\rho+p}{\rho+3p}\,\frac{d\rho}{dp}$. One can check that it is for example satisfied for the equation of state of a relativistic ideal Fermi gas at zero temperature, but only up to densities of order $10^{15}\mathrm{gcm}^{-3}$, which is roughly the critical density where gravitational instability sets in. It is known from numerical results [69] that beyond that limit the uniqueness statement of the above theorem will fail. One believes however, that sphericity will still hold.

We will here confine ourselves to an outline of the proof to the case of the special equation of state given by [13]

$$\rho(p) = \frac{1}{6}\,\rho^{6/5}(\rho_0^{1/5} - \rho^{1/5})^{-1} \qquad (\rho_0 = \mathrm{const} > 0) \tag{5.10}$$

which is a relativistic generalization of the equation for a polytrope of index 5 in the Newtonian theory . The expression in (5.10) satisfies $I \equiv 0$. The reference spherical solution in this case is known explicitly [7]. It has the curious property that it is asymptotically flat, but the fluid extends to spatial infinity.

Introducing the variable $v = (-V)^{1/2}$, the static field equations for a perfect fluid with energy momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{5.11}$$

with $u_\mu = v^{-1}\xi_\mu$ read

$$D^2 v = 4\pi v(\rho + 3p) \tag{5.12}$$

$$\mathcal{R}_{ij} = v^{-1} D_i D_j v + 4\pi(\rho - p)h_{ij} \tag{5.13}$$

The asymptotic conditions (2.102,104) imply that $v \to 1$ at infinity. ¿From the maximum principle for elliptic equations it follows that $0 < v < 1$ in N. Since the surface of the star is at infinity for the Buchdahl solution, the $v|_{\partial S}$, which is always equal to one in that case, has to be replaced by the total mass M (see Sect. 4.1).

Applying the contracted Bianchi identity to (5.12,13), there follows

$$D_i p = -v^{-1}(\rho + p) D_i v. \tag{5.14}$$

Thus p and ρ are both functions of v and

$$\frac{dp}{dv} = -v^{-1}(\rho + p). \tag{5.15}$$

Define the Cotton tensor of h_{ij} B_{ijk} by

$$B_{ijk} = 2D_{[k}\left(\mathcal{R}_{j]i} - \frac{1}{2}h_{j]i}\mathcal{R}\right). \tag{5.16}$$

With the definition

$$W = (D_i v)(D^i v) \tag{5.17}$$

the equations (5.12,13) now imply (see Lindblom [46]) that

$$\begin{aligned} D^2 W = {} & \frac{1}{4}v^4 W^{-1} B_{ijk}B^{ijk} + v^{-1}(D^i v)(D_i W) + 8\pi v (D^i v)(D_i \rho) + \frac{3}{4}W^{-1}(D^i W)(D_i W) \\ & - 8\pi W(\rho + p) + 16\pi^2 v^2 (\rho + 3p)^2 - 4\pi v(\rho + 3p) W^{-1}(D^i v)(D_i W). \end{aligned} \tag{5.18}$$

In the spherically symmetric case $W = W_0$ has to be of the form $W_0 = W_0(v)$. The ODE resulting in that case from (5.18), has, for the equation of state (5.10), an explicit solution given by

$$W_0 = (1 - v^2)^4 \left[\frac{1}{16M^2} - \frac{\pi\rho_0}{3}\left(\frac{1-v}{1+v}\right)^2\right]. \tag{5.19}$$

We assume that $\alpha = \frac{16\pi}{3}\rho_0 M^2 > 1$. The function W_0 is defined for $v \in [0, 1]$. It is positive for $v \in (v_c, 1)$, with $v_c = (\sqrt{\alpha} - 1)/(\sqrt{\alpha} + 1)$ and $W_0(v_c) = 0$, $W_0(1) = 0$. Thus W_0 satisfies the correct boundary condition at the central value v_c of v and at infinity.

We now define, for the given solution (v, h_{ij}), the scalar function

$$\widetilde{W} - \widetilde{W}_0 = \left(\frac{1 - v^2}{2}\right)^{-4}(W - W_0) \tag{5.20}$$

and the conformally rescaled metric

$$\widetilde{h}_{ij} = v^{-2}\left(\frac{1 - v^2}{2}\right)^4 h_{ij}. \tag{5.21}$$

(The constant M occurring in W_0 is taken to be the mass of the given solution.) In the asymptotically flat, vacuum case discussed in Sect. 3.2 one finds that the metric $\widetilde{h}_{ij}$ extends smoothly to the manifold $\widetilde{N} = N \cup \{\Lambda\}$, with Λ the point at infinity. This is also true for the Buchdahl solution, and we assume it to be true for the given, a priori non-spherical one. After some calculations we find that

$$\widetilde{D}^2(\widetilde{W} - \widetilde{W}_0) = \frac{1}{4}\widetilde{W}^4 \widetilde{B}_{ijk}\widetilde{B}^{ijk} + \frac{3}{4}\widetilde{W}^{-1}\widetilde{D}^i(\widetilde{W} - \widetilde{W}_0)\widetilde{D}_i(\widetilde{W} - \widetilde{W}_0). \quad (5.22)$$

Since $\widetilde{W}, \widetilde{W}_0$ also extend smoothly to $\widetilde{N}$, the function $\widetilde{W} - \widetilde{W}_0$ satisfies the elliptic equation with nonnegative right-hand side on the compact manifold $\widetilde{N}$. After integrating (5.22) over $\widetilde{N}$ (or by the maximum principle) it follows that $\widetilde{B}_{ijk}$ is zero and

$$\widetilde{W} = \widetilde{W}_0(v). \quad (5.23)$$

It then follows from [8], that the given model is isometric to the Buchdahl solution with the same ρ_0 and the same M.

5.3 Spherically Symmetric, Static Perfect Fluid Solutions

The metric for a static spherically symmetric spacetime can be written

$$ds^2 = -c^2 e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2) \; . \quad (5.24)$$

For a derivation see [28,70]. Here c is a constant which plays the role of the speed of light. In Appendix B of [28] it is also demonstrated that the r^2 in front of the sphere metric is no loss of generality for a static perfect fluid with positive mass density and pressure. Hence it is impossible to have two centers or two infinities. The field equations for a perfect fluid are

$$8\pi G c^{-2}\rho r^2 = e^{-\lambda}(r\lambda' - 1) + 1 \quad (5.25)$$

$$8\pi G c^{-4} p r^2 = e^{-\lambda}(r\nu' + 1) - 1 \quad (5.26)$$

$$8\pi G c^{-4} p = \frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{1}{2}\nu'^2 + r^{-1}(\nu' - \lambda') - \frac{1}{2}\nu'\lambda'\right) \quad (5.27)$$

A prime denotes a derivative with respect to r. We have written $-c^2\rho$ for the timelike eigenvector of the energy–momentum tensor to make the comparison with the Newtonian equations easier. The field equation imply 'energy–momentum conservation', which is a single equation for a static perfect fluid

$$2p' = -\nu'(p + c^2\rho) \; . \quad (5.28)$$

The first exact solution of these equation was alredy found in 1918 by Karl Schwarzschild, the solution with constant density [37]. We have three independent ordinary differential equations for for four functions. Hence, one function can be specified freely. The most physical case is to prescribe an equation of state $\rho = \rho(p)$. Equation (5.25) can easily be integrated:

$$e^{-\lambda} = 1 - \frac{8\pi G}{c^2}\frac{1}{r}\int r^2\rho(r)dr + \mathrm{const} \; . \quad (5.29)$$

As we only are interested in solutions with a regular center of spherical symmetry we define λ as follows

$$e^{-\lambda} = 1 - \frac{8\pi G}{c^2}\frac{1}{r}\int_0^r r^2\rho(r)dr \ . \tag{5.30}$$

The usual definition of the 'mass up to r', namely

$$m(r) = 4\pi\int_0^r r^2\rho(s)ds \tag{5.31}$$

gives

$$e^{-\lambda} = 1 - \frac{2G}{c^2}\frac{m(r)}{r} \ . \tag{5.32}$$

It is also useful to introduce the following quantity which is related to the 'mean density up to r'

$$w(r) = r^{-3}m(r) \ . \tag{5.33}$$

Then (5.32) becomes

$$e^{-\lambda} = 1 - \frac{2G}{c^2}r^2 w \ . \tag{5.34}$$

Various forms of the equations (5.25-28) will be used. Equations (5.25), (5.26) and (5.28) contain all the information. If we eliminate ν' then (5.26) and (5.28) imply the Tolman – Oppenheimer – Volkoff equation [75]

$$p' = -Gr\left(1 - \frac{2G}{c^2}r^2 w\right)^{-1}\left(\frac{4\pi p}{c^2} + w\right)\left(\frac{p}{c^2} + \rho\right) \tag{5.35}$$

If an equation of state is given we can integrate (5.28)

$$\nu(r) = -\int_{p_0}^{p(r)} \frac{2dp}{p + c^2\rho(p)} + \text{constant} \tag{5.36}$$

In this formula p_0 denotes the central pressure. If we add the definition of w then (5.35) and (5.33) form an integro–differential system. Differentiating (5.33) we obtain

$$w' = \frac{1}{r}(4\pi\rho - 3w) \tag{5.37}$$

In [65] the following theorem is proved.

Theorem: Let an equation of state $\rho(p)$ be given such that ρ is defined for $p \geq 0$, non–negative and continuous for $p \geq 0$, C^∞ for $p > 0$ and suppose that $d\rho/dp > 0$ for $p > 0$.

Then there exists for any value of the central density ρ_0 a unique inextensible, static, spherically symmetric solution of Einstein's field equation with a perfect fluid source and equation of state $\rho(p)$. The matter either has finite extent, in which case a unique Schwarzschild solution is joined on as an exterior field, or the matter occupies the whole space, with ρ tending to 0 as r tends to infinity.

There are two parts of the proof. The equations (5.35) and (5.37) form a system of ordinary differential equations for $p(r), w(r)$. However, the system is singular at $r =$ and the first step is to demonstrate that for each value of the central density there is a unique solution such that the spacetime is regular at the center. This is shown in [65] or in [50]. This solution defines a neighborhood of a regular center and can be extended as long as $(1 - \frac{2G}{c^2} r^2 w)$ remains positive. This can be seen as follows.

Introduce the variables first used by Buchdahl [13]

$$y^2 = 1 - \frac{2G}{c^2} r^2 w \ , \qquad \zeta = e^{\nu/2} \ , \qquad x = r^2 \tag{5.38}$$

Rewriting the equations in these variables and eliminating p in 5.26) and (5.28) gives an equation which is linear in ζ and w,

$$(1 - \frac{2G}{c^2} x w) \zeta_{,xx} - \frac{G}{c^2} \zeta_{,x} (w + x w_{,x})_{,x} - \frac{G}{2c^2} w_{,x} \zeta = 0 \tag{5.39}$$

or

$$(y \zeta_{,x})_{,x} - \frac{G}{2c^2} \frac{w_{,x} \zeta}{y} = 0 \tag{5.40}$$

Let $0 \leq x < x_0$ be an intervall such that $y^2 = (1 - \frac{2G}{c^2} x w > 0$ and $p > 0$. As the density does not increase outwards we have $w_{,x} \leq 0$. Therefore

$$(y \zeta_{,x})_{,x} \leq 0 \tag{5.41}$$

The equation (5.26) can be rewritten as

$$y \zeta_{,x} = \frac{\zeta}{y} \frac{G}{2c^2} (w + \frac{4\pi}{c^2} p) \ . \tag{5.42}$$

¿From (5.41) and (5.42) we obtain the inequality

$$y \geq \frac{w + 4\pi p / c^2}{w_0 + 4\pi p_0 / c^2} \ . \tag{5.43}$$

Hence, we see that y cannot vanish before p.

Suppose $p(x_b) = 0$. Then we call the corresponding r_b the radius of the star. The Schwarzschild solution is given in the form $e^{-\lambda} = e^{\nu} = 1 - A/r$ for some constant A. Hence, we determine a unique exterior field by the condition $A = (\frac{2G}{c^2}) m(r_b)$. In this way the matter solution and the outside solution are joined only in a C^0–fashion because the boundary density may be non–zero. If we introduce Gauss coordinates relative to the hypersurface $p = 0$ the metric is C^1. It is obvious that this metric cannot be extended because the area of the group orbits $r =$ constant grows from 0 to infinity.

Let us now consider the second possibility that $p(x) > 0$ for all x. Because $p(x)$ is monotonically decreasing for $x \to \infty$, $\lim_{x\to\infty} p(x) = p_\infty$ exists. This implies that p' tends to 0 for $x \to \infty$. Since $y \leq 1$, (5.35) then implies $p_\infty = 0$ and hence, using the equation of state, that $\rho \to 0$ as $x \to \infty$. As before the spacetime is not extensible.

This completes our outline of the proof. It shows in particular that for for $\rho(p)$ with $\rho(0) = \rho_b > 0$ the radius of the star has to be finite.

There are various exact global solutions known. (For a useful list of such solutions including a discussion of their physical acceptability has been given by Delgaty and Lake [19].) For the 1–parameter family of equations of state given by Equ. (5.10) the whole 1–parameter family of solutions is known. A 2–parameter family of equations of state of interest for the issue of section 3.1 is investigated by Simon [72]; all the corresponding exact solutions are given.

There are some conditions on the equation of state known, which allow to decide whether the radius of the star is finite or infinite in the case of vanishing boundary density ρ_b. In [65] it is shown that the radius of the star is finite if $\int_o^{p_0} dp/\rho(p)^2$ is finite. Conversely, $\int_o^{p_0} dp/(\rho(p)c^{-2}p < \infty$ implies that the matter distribution is infinitely extended. Both conditions depend only on the behaviour of the equation of state near the boundary $p = 0$. Makino [50] gives conditions for a finite radius in cases which are not covered by the above. He shows in particular, that for polytropic equations of state, $p = const.\rho^\gamma$ with $4/3 < \gamma < 2$ the radius is finite.

For finite distributions "Buchdahl's inequality" holds [13].

Theorem: For finite distributions with non–negative density and a monotonic equations of state there holds

$$1 - \frac{2G}{c^2}\frac{M}{r_b} > \frac{1}{9}\ . \tag{5.44}$$

Proof: To obtain the inequality one compares the solution with a solution of constant density ρ, an interior Schwarzschild solution. Equ. (5.40) implies for this solution (written with an overbar) that

$$(\bar{y}\bar{\zeta}_{,x})_{,x} = 0 \quad\Longrightarrow \bar{y}\bar{\zeta}_{,x} = a = \text{constant} \tag{5.45}$$

We normalize ζ by the condition that at the boundary we have $\zeta_b = y_b$. Then we find a if we rewrite (5.26) in the new varables

$$\frac{8\pi G}{c^2}p = 4y^2\frac{\zeta_{,x}}{\zeta} - \frac{2G}{c^2}w \tag{5.46}$$

and evaluate it at the boundary as $a = \frac{2G}{c^2}\bar{w}$.

Then (5.45) can be integrated with the result

$$\bar{\zeta}(x) = \frac{1}{2}\left(1 + 2\bar{\zeta}(0) + \sqrt{1 - \frac{2G}{c^2}x\bar{w}}\right) \tag{5.47}$$

Now (5.41) implies

$$y\zeta_{,x} > (y\zeta_{,x})_b = \bar{y}\bar{\zeta}_{,x} \tag{5.48}$$

As $\bar{y} > y$ we obtain

$$\zeta_{,x} \geq \bar{\zeta}_{,x} = \frac{1}{2}\left(1 + 2\bar{\zeta}(0) + \sqrt{1 - \frac{2G}{c^2} x\bar{w}}\right) \tag{5.49}$$

As $\bar{\zeta}$ is positive we obtain at the boundary

$$y_b \geq -\frac{1}{2} y_b + \frac{1}{2} \tag{5.50}$$

which is (5.44).

Buchdahl's inequality show that one can pack only a certain mass into a given fixed radius. The physical reason is that the pressure is also a source of the gravitational field. In Newton's theory there are constant density balls with a fixed radius for arbitrary density. In Einstein's theory the central pressure diverges if the density approaches some maximum value.

In [1] an analogue of Buchdahls inequality is derived for distributions in which the the density is only assumed to be positive. There holds $1 - \frac{2G}{c^2}\frac{M}{r_b} > 0$.

Another important topic are bounds on the total mass of the system. Suppose we know the equation of state only for $\rho < \rho_0$. Then we can estimate the mass and radius of a core in which the density is greater ρ_0 as follows: Clearly, $m(r_0) > \frac{4\pi}{3}\rho_0(r_0)^3$; because of $y > 0$ we have also $m(r_0) < \frac{c^2}{2G} r_0$. Hence the possible cores occupy a compact part of the $m(r_0)$–r_0– plane. Taking intial values from this part one can numerically integrate outwards using the known equation of state, until the pressure vanishes. This was done in [27] for $\rho_0 = 5.1 \times 10^{14} g/cm^3$ and with a certain realistic equation of state for smaller densities. All configurations had a total mass smaller then $5M_\odot$. It is quite remarkable the the knowledge of the equation of state for a finite density range allows to show such a bound on the total mass, assuming nothing but the monotonicity of the equation of state in the unknown density range. This is not possible in Newton's theory.

In the special case of bodies with a sharp edge, i.e $\rho_b > 0$, we can combine the Buchdahl inequality (5.44) with the estimate $M = m(r_b) \geq 4\pi\rho_b r_b^3$ to obtain the mass bound

$$M \leq \left(\frac{2}{3}\right)^3 \left(\frac{3c^6}{4\pi G^3 \rho_b}\right)^{1/2} . \tag{5.51}$$

Let us finally compare with Newton's theory. In (5.35) it is almost obvious that for $c \to \infty$ one obtains the Newtonian equation for the pressure. The relativistic corrections show how "the pressure enters in the active and passive gravitational mass". The first factor describes an effect of the geometry. Static fluid ball are the simplest examples of families of relativistic solutions with a Newtonian limit [21].

5.4 Spherically Symmetric, Static Einstein–Vlasov Solutions

In recent years existence and further properties of solutions of Einstein's field equations for a collisionless gas have been shown [64]. The Vlasov–Einstein– system determines the spacetime metric and the distribution function $f(x^\mu, p^\mu)$ describing the particles.

$$\begin{aligned} p^\mu \partial_{x^\mu} f - \Gamma^\mu_{\nu\sigma} p^\nu p^\sigma \partial_{p^\mu} f &= 0 \\ T^{\mu\nu} &= \int p^\mu p^\nu |g|^{1/2} \frac{d^4 p}{m} \\ G^{\mu\nu} &= 8\pi T^{\mu\nu} . \end{aligned} \tag{5.52}$$

In the static spherically symmetric case and for the metric (5.24), these equations reduce to (r= $|x^i|$, v^i are the spatial frame components of p^α)

$$\frac{v^i}{\sqrt{1+v^2}} \partial_{x^i} f - \sqrt{1+v^2}\, \nu' \frac{x^i}{r} \partial_{v^i} f = 0 \tag{5.53}$$

$$8\pi G c^{-2} \rho r^2 = e^{-\lambda}(r\lambda' - 1) + 1 \tag{5.54}$$

$$8\pi G c^{-4} p r^2 = e^{-\lambda}(r\nu' + 1) - 1 \tag{5.55}$$

where

$$\rho(x) = \rho(r) = \int_{R^3} f(x^i, v^i) \sqrt{1+v^2}\, dv \ , \tag{5.56}$$

$$p(x) = p(r) = \int_{R^3} f(x^i, v^i) \left(\frac{x^i v_i}{r} \right) \frac{dv}{\sqrt{1+v^2}} \ . \tag{5.57}$$

The distribution function is assumed to be spherically symmetric.

Rein and Rendall [64] show the existence of asymptotically flat solutions, regular at the center, with finite total mass and finite extension of the matter and isotropic pressure. It is also possible to construct solutions with anisotropic pressure; Furthermore shells of finite extent of matter around a regular center or a black hole can be constructed [63].

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